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Distributed MPC Strategies with Application to Power System Automatic Generation Control[*]

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Abstract

A distributed model predictive control (MPC) framework, suitable for controlling large-scale networked systems such as power systems, is presented. The overall system is decomposed into subsystems, each with its own MPC controller. These subsystem-based MPCs work iteratively and cooperatively towards satisfying systemwide control objectives. If available computational time allows convergence, the proposed distributed MPC framework achieves performance equivalent to centralized MPC. Furthermore, the distributed MPC algorithm is feasible and closed-loop stable under intermediate termination. Automatic generation control (AGC) provides a practical example for illustrating the efficacy of the proposed distributed MPC framework.

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1 Introduction

Model predictive control (MPC) is widely recognized as a high performance, yet practical, control technology. This model-based control strategy uses a prediction of system response to establish an appropriate control response. An attractive attribute of MPC technology is its ability to systematically account for process constraints. The effectiveness of MPC is dependent on a model of acceptable accuracy and the availability of sufficiently fast computational resources. These requirements limit the application base for MPC. Even so, applications abound in the process industries, and are becoming more widespread [5, 24].

Traditionally, control of large, networked systems is achieved by designing local, subsystem-based controllers that ignore the interactions between the different subsystems. A survey of decentralized control methods for large-scale systems is available in [25]. It is well known that a decentralized control philosophy may result in poor systemwide control performance if the subsystems interact significantly. Centralized MPC, on the other hand, is impractical for control of large-scale, geographically expansive systems, such as power systems. A distributed MPC framework is appealing in this context; the distributed MPC controllers must, however, account for the interactions between the subsystems. These and other issues critical to the success of distributed MPC are examined in this paper.

Each MPC, in addition to determining the optimal current response, also generates a prediction of future subsystem behavior. By suitably leveraging this prediction of future subsystem behavior, the various subsystem-based MPCs can be integrated and the overall system performance improved. A discussion on economic and performance benefits attainable by integrating subsystem-based MPCs is available in [15, 21]. One of the goals of this paper, however, is to illustrate that a simple exchange of predicted subsystem trajectories (*communication*) does not necessarily improve overall system control performance.

A few distributed MPC formulations are available in the literature. A distributed MPC framework was proposed in [11], for the class of systems that have independent subsystem dynamics but are linked through their cost functions. More recently in [10], an extension of the method described in [11] that handles systems with weakly interacting subsystem dynamics was proposed. Stability is proved through the use of a conservative, consistency constraint that forces the predicted and assumed input trajectories to be close to each other. Also, as pointed out by the author, the stability analysis in [10] requires the number of agents to be at least 10; this requirement on the minimum number of agents raises concerns on the applicability of the distributed MPC framework proposed in [10] in a general setting. Furthermore, the performance of the distributed MPC framework in [10] is different from that of centralized MPC. A distributed MPC algorithm for unconstrained, linear time-invariant (LTI) systems was proposed in [6, 18]. For the models considered in [6, 18], the evolution of the states of each subsystem is assumed to be influenced only by the states of interacting subsystems and local subsystem inputs. This choice of modeling framework can be restrictive. In many cases, such as the two area power network with FACTS device (Section 5.7.5) and most chemical plants, the evolution of the subsystem states is also influenced by the inputs of interconnected subsystems. More crucially for the distributed MPC framework proposed in [6, 18], the subsystem-based MPCs have no knowledge of each other's cost/utility functions. It is known from noncooperative game theory that if such pure *communication-based strategies* (in which competing agents have no knowledge of each others cost functions) converge, they converge to the Nash equilibrium (NE) ([1, 2]). In most cases involving a finite number of agents, the NE is different from the Pareto optimal (PO) solution [8, 9, 23]. In fact, nonconvergence or suboptimality of pure communication-based strategies may result in unstable closed-loop behavior in some cases. A four area power network example is used here (Section 5.7.4) to illustrate instability due to communication-based MPC. Such examples are not uncommon. A distributed MPC framework in which the effect of interconnected subsystems are treated as bounded uncertainties was proposed in [19]. Stability and optimality properties have not been established however.

Most interconnected power systems rely on automatic generation control (AGC) for controlling system frequency and tie-line interchange [31]. These objectives are achieved by regulating the real power output of generators throughout the system. To cope with the expansive nature of power systems, a distributed control structure has been adopted for AGC. Also, various limits must be taken into account, including restrictions on the amount and rate of generator power deviation. AGC therefore provides a very relevant example for illustrating the performance of distributed MPC in a power system setting.

Flexible AC transmission system (FACTS) devices allow control of the real power flow over selected paths through a transmission network [16] . As transmission systems become more heavily loaded, such controllability offers economic benefits [20] . However FACTS controls must be coordinated with each other, and with other power system controls, including AGC. Distributed MPC offers an effective means of achieving such coordination, whilst alleviating the organizational and computational burden associated with centralized control.

This paper is organized as follows. In Section 2, a brief description of the different modeling frameworks is presented. Notation used in this paper is introduced in Section 3. In Section 4, a description of the different MPC based systemwide control frameworks is provided. A simple example that illustrates the unreliability of communication-based MPC is presented. An implementable algorithm for terminal penalty-based distributed MPC is described in Section 5. Properties of this distributed MPC algorithm and closed-loop properties of the resulting distributed controller are established subsequently. Three examples are presented to assess the performance of the terminal penalty-based distributed MPC framework. Two useful extensions of the proposed distributed MPC framework are described in Section 6. An algorithm for terminal control-based distributed MPC is described in Section 7. Two examples are presented to illustrate the efficacy of the terminal control-based distributed MPC framework. Conclusions of this study are provided in Section 8. Some supporting material and proofs of results in the paper are available in Appendices A- B.

2 Models

Distributed MPC relies on decomposing the overall system model into appropriate subsystem models. A system comprised of M interconnected subsystems will be used to establish these concepts.

Centralized model. The overall system model is represented as a discrete, linear time-invariant (LTI) model of the form

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

in which k denotes discrete time and

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1M} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i1} & A_{i2} & \dots & A_{iM} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \dots & A_{MM} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1M} \\ \vdots & \vdots & \ddots & \vdots \\ B_{i1} & B_{i2} & \dots & B_{iM} \\ \vdots & \vdots & \ddots & \vdots \\ B_{M1} & B_{M2} & \dots & B_{MM} \end{bmatrix}$$

$$C = \begin{bmatrix} C_{11} & 0 & \dots & 0 \\ 0 & C_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & C_{MM} \end{bmatrix} \quad u = \begin{bmatrix} u_1{}' & u_2{}' & \dots & u_M{}' \end{bmatrix}' \in \mathbb{R}^m$$

$$x = \begin{bmatrix} x_1{}' & x_2{}' & \dots & x_M{}' \end{bmatrix}' \in \mathbb{R}^n \quad y = \begin{bmatrix} y_1{}' & y_2{}' & \dots & y_M{}' \end{bmatrix}' \in \mathbb{R}^z.$$

For each subsystem $i = 1, 2, \dots, M$, the triplet (u_i, x_i, y_i) represents the subsystem input, state and output vector respectively. The centralized model pair (A, B) is assumed to be stabilizable and (A, C) is detectable [1].

Decentralized model. In the decentralized modeling framework, it is assumed that the interaction between the subsystems is negligible. Subsequently, the effect of the external subsystems on the local subsystem is ignored in this modeling framework. The decentralized model for subsystem $i = 1, 2, \dots, M$ is

$$\begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + B_{ii}u_i(k) \\ y_i(k) &= C_{ii}x_i(k) \end{aligned}$$

Partitioned model (PM). The PM for subsystem i combines the effect of the local subsystem variables and the effect of the states and inputs of the interconnected subsystems. The PM for subsystem i is obtained by considering the relevant partition of the centralized model and can be explicitly written as

$$x_i(k+1) = A_{ii}x_i(k) + B_{ii}u_i(k) + \sum_{j \neq i} (A_{ij}x_j(k) + B_{ij}u_j(k)) \tag{1a}$$

$$y_i(k) = C_{ii}x_i(k) \tag{1b}$$

[1] In the applications considered here, local measurements are typically a subset of subsystem states. The structure selected for the C matrix reflects this observation. A general C matrix may be used, but impacts possible choices for distributed estimation techniques [29].

3 Notation

For any matrix P, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum (absolute) eigenvalue of P respectively. For any subsystem $i = 1, 2, \ldots, M$, let the predicted state and input at time instant $k+j,\ j \geq 0$, based on data at time k be denoted by $x_i(k+j|k) \in \mathbb{R}^{n_i}$ and $u_i(k+j|k) \in \mathbb{R}^{m_i}$ respectively. We have the following definitions for the infinite horizon predicted state and input trajectory vectors in the different MPC frameworks

$$\begin{aligned}
\text{Centralized state trajectory}: \boldsymbol{x}(k)' &= \left[x(k+1|k)', x(k+2|k)', \ldots\ldots\right] \\
\text{Centralized input trajectory}: \boldsymbol{u}(k)' &= \left[u(k|k)', u(k+1|k)', \ldots\ldots\right] \\
\text{State trajectory (subsystem } i): \boldsymbol{x}_i(k)' &= \left[x_i(k+1|k)', x_i(k+2|k)', \ldots\ldots\right] \\
\text{Input trajectory (subsystem } i): \boldsymbol{u}_i(k)' &= \left[u_i(k|k)', u_i(k+1|k)', \ldots\ldots\right]
\end{aligned}$$

Let N denote the control horizon. The following notation is used to represent the finite horizon predicted state and input trajectory vectors in the different MPC frameworks

$$\begin{aligned}
\text{Centralized state trajectory}: \overline{\boldsymbol{x}}(k)' &= \left[x(k+1|k)', x(k+2|k)', \ldots, x(k+N|k)'\right] \\
\text{Centralized input trajectory}: \overline{\boldsymbol{u}}(k)' &= \left[u(k|k)', u(k+1|k)', \ldots, u(k+N-1|k)'\right] \\
\text{State trajectory (subsystem } i): \overline{\boldsymbol{x}}_i(k)' &= \left[x_i(k+1|k)', x_i(k+2|k)', \ldots, x_i(k+N|k)'\right] \\
\text{Input trajectory (subsystem } i): \overline{\boldsymbol{u}}_i(k)' &= \left[u_i(k|k)', u_i(k+1|k)', \ldots, u_i(k+N-1|k)'\right]
\end{aligned}$$

4 MPC frameworks for systemwide control

The set of admissible controls for subsystem $i, \Omega_i \subseteq \mathbb{R}^{m_i}$ is assumed to be a nonempty, compact, convex set with $0 \in \text{int}(\Omega_i)$. The set of admissible controls for the whole plant Ω is defined to be the Cartesian product of the admissible control sets $\Omega_i, \forall\, i = 1, 2, \ldots, M$.

The *stage cost* at stage $t \geq k$ along the prediction horizon is defined as

$$L_i(x_i(t|k), u_i(t|k)) = \frac{1}{2}\left[x_i(t|k)' Q_i x_i(t|k) + u_i(t|k)' R_i u_i(t|k)\right] \tag{2}$$

in which $Q_i \geq 0$, $R_i > 0$ are symmetric weighting matrices and $(A_i, Q_i^{1/2})$ is detectable. The *cost function* $\phi_i(\cdot)$ for subsystem i is defined over an infinite horizon and is written as

$$\phi_i\left(\boldsymbol{x}_i, \boldsymbol{u}_i; x_i(k)\right) = \sum_{t=k}^{\infty} L_i\left(x_i(t|k), u_i(t|k)\right) \tag{3}$$

with $x_i(k|k) \equiv x_i(k)$. For any system, the constrained stabilizable set (also termed Null controllable domain) $\mathcal{X}$ is the set of all initial states $x \subseteq \mathbb{R}^n$ that can be steered to the origin by applying a sequence of admissible controls (see [28, Definition 2]). It is assumed throughout that the initial system state vector $x(k) \in \mathcal{X}$, in which $\mathcal{X}$ denotes the constrained stabilizable set for the overall system. A feasible solution to the corresponding optimization problem, therefore, exists. For notational simplicity, we drop the time dependence of the state and input trajectories in each MPC framework. For instance in the centralized MPC framework, we write

$\boldsymbol{x} \leftarrow \boldsymbol{x}(k)$ and $\boldsymbol{u} \leftarrow \boldsymbol{u}(k)$. In the distributed MPC framework, we use $\boldsymbol{x}_i \leftarrow \boldsymbol{x}_i(k)$ and $\boldsymbol{u}_i \leftarrow \boldsymbol{u}_i(k),\ \forall\, i = 1, 2, \ldots M$.

Four MPC based systemwide control frameworks are described below. In each MPC framework, the controller is defined by implementing the first input in the solution to the corresponding optimization problem.

Centralized MPC. In the centralized MPC framework, the MPC for the overall system solves the following optimization problem

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{u}} \quad & \phi\left(\boldsymbol{x}, \boldsymbol{u}; x(k)\right) = \sum_i w_i \phi_i\left(\boldsymbol{x}_i, \boldsymbol{u}_i; x_i(k)\right) \\ \text{subject to} & \\ x(l+1|k) &= Ax(l|k) + Bu(l|k), \quad k \le l \\ u_i(l|k) &\in \Omega_i, \quad k \le l, \quad i = 1, 2, \ldots, M \end{aligned}$$

where $w_i > 0, \sum w_i = 1$.

For any system, centralized MPC achieves the best attainable performance (Pareto optimal) as the effect of interconnections among subsystems are accounted for exactly. Furthermore, any conflicts among controller objectives are resolved optimally.

Decentralized MPC. In the decentralized MPC framework, each subsystem-based MPC solves the following optimization problem

$$\begin{aligned} \min_{\boldsymbol{x}_i,\boldsymbol{u}_i} \quad & \phi_i\left(\boldsymbol{x}_i, \boldsymbol{u}_i; x_i(k)\right) \\ \text{subject to} & \\ x_i(l+1|k) &= A_{ii}x_i(l|k) + B_{ii}u_i(l|k), \quad k \le l \\ u_i(l|k) &\in \Omega_i, \quad k \le l \end{aligned}$$

Each decentralized MPC solves an optimization problem to minimize its (local) cost function. The effects of the interconnected subsystems are assumed to be negligible and are ignored. In many situations, however, the above assumption is not valid and leads to reduced control performance.

Distributed MPC. The partitioned model for each subsystem $i = 1, 2, \ldots, M$ is assumed to be available. Two formulations for distributed MPC namely, communication-based MPC and cooperation-based MPC, are considered. Distributed MPC formulations based on pure communication based strategies are available in the literature [6, 18]. In the sequel, the suitability of pure communication based MPC, as a candidate systemwide control formulation, is assessed. For both communication and cooperation-based MPC, several subsystem optimizations and exchange of variables between subsystems are performed during a sample time. An optimization and exchanges of variables is termed an *iterate*. We may choose not to iterate to convergence. The iteration number is denoted by p.

Communication-based MPC. For communication-based MPC [2], the optimal state-input trajectory $(\boldsymbol{x}_i^p, \boldsymbol{u}_i^p)$ for subsystem i, $i = 1, 2, \ldots, M$ at iterate p is obtained as the solution to the optimization problem

$$\min_{\boldsymbol{x}_i, \boldsymbol{u}_i} \quad \phi_i\left(\boldsymbol{x}_i, \boldsymbol{u}_i; x_i(k)\right)$$

subject to

$$x_i(l+1|k) = A_{ii}x_i(l|k) + B_{ii}u_i(l|k) + \sum_{j \neq i}[A_{ij}x_j^{p-1}(l|k) + B_{ij}u_j^{p-1}(l|k)],\ k \leq l$$

$$u_i(l|k) \in \Omega_i,\ k \leq l$$

Each communication-based MPC utilizes the objective function for that subsystem only. For each subsystem i at iteration p, only that subsystem input sequence $\boldsymbol{u}_i$ is optimized and updated. The other subsystems' inputs remain at $\boldsymbol{u}_j^{p-1}, \forall\ j = 1, 2, \ldots, M, j \neq i$. If the communication-based iterates converge, then at convergence, the Nash equilibrium (NE) is achieved. In this work, the term *communication-based MPC* alludes to the above framework at convergence of the exchanged trajectories.

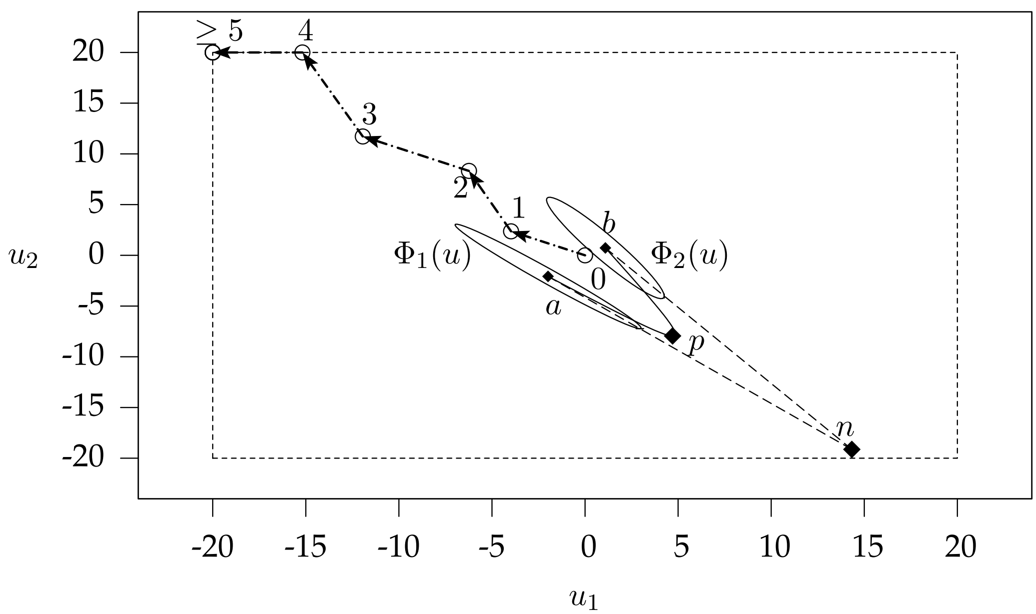

Figure 1: A Nash equilibrium exists. Communication-based iterates, however, do not converge to the Nash equilibrium.

Instability under communication-based MPC. Figure 1 illustrates nonconvergence of communication-based MPC for a two subsystem case. The details of the example are omitted here for brevity. For initial values of inputs at the origin and in the absence of input constraints, the sequence of communication-based iterates diverges to infinity. For a compact feasible region (the

[2]Similar strategies have been proposed by [6, 18]

box in Figure 1), the sequence of communication-based iterates is trapped at the boundary of the feasible region (Point 5). For this system, the NE is at point n.

In the communication-based MPC framework, each subsystem's MPC has no information about the objectives of the interconnected subsystems' MPCs. Convergence of the exchanged state and input trajectories is implicitly assumed and is, therefore, a drawback of this formulation. In many cases, the NE cannot be achieved using pure communication-based strategies [2]. Even in cases where convergence to the NE is achieved, pure communication-based strategies cannot guarantee closed-loop stability. Communication-based MPC is, therefore, an unreliable strategy for systemwide control.

Feasible cooperation-based MPC (FC-MPC). To arrive at a reliable distributed MPC framework, we need to ensure that the subsystems' MPCs cooperate, rather than compete, with each other in achieving systemwide objectives. The local controller objective $\phi_i(\cdot)$ is replaced by an objective that measures the systemwide impact of local control actions. The simplest choice for such an objective is a strict convex combination of the controller objectives $i.e.,\ \phi(\cdot) = \sum_i w_i\phi_i(\cdot),\ w_i > 0,\ \sum_i w_i = 1$.

In large-scale implementations, the system sampling interval may be insufficient to allow convergence of an iterative, cooperation-based algorithm. In such cases, the cooperation-based algorithm has to be terminated prior to convergence of exchanged trajectories. The final calculated input trajectories are used to define a suitable distributed MPC control law. To enable intermediate termination, it is necessary that all iterates generated by the cooperation-based algorithm are strictly *systemwide feasible* (*i.e.*, satisfy all model and inequality constraints) and the resulting nominal distributed control law is closed-loop stable. Such a distributed MPC algorithm is presented in Section 5.

For notational convenience, we drop the k dependence of $\overline{\boldsymbol{x}}_i(k), \overline{\boldsymbol{u}}_i(k), i = 1, 2, \ldots, M$. It is shown in Appendix A that each $\overline{\boldsymbol{x}}_i$ can be expressed as

$$\overline{\boldsymbol{x}}_i = E_{ii}\overline{\boldsymbol{u}}_i + f_{ii}x_i(k) + \sum_{j\neq i}[E_{ij}\overline{\boldsymbol{u}}_j + f_{ij}x_j(k)]. \tag{6}$$

We consider open-loop stable systems here. Extensions of the distributed MPC methodology to handle large, open-loop unstable systems are described in Sections 6.2 and 7.

For open-loop stable systems, the FC-MPC optimization problem for subsystem i, denoted $\mathcal{F}_i$, is defined as

$$\mathcal{F}_i \triangleq \min_{\boldsymbol{u}_i} \quad \sum_{r=1}^{M} w_r \Phi_r \left(\boldsymbol{u}_1^{p-1}, \ldots, \boldsymbol{u}_{i-1}^{p-1}, \boldsymbol{u}_i, \boldsymbol{u}_{i+1}^{p-1}, \ldots, \boldsymbol{u}_M^{p-1}; x_r(k)\right) \tag{7a}$$

subject to

$$u_i(t|k) \in \Omega_i, \quad k \leq t \leq k + N - 1 \tag{7b}$$

$$u_i(t|k) = 0, \quad k + N \leq t \tag{7c}$$

The infinite horizon input trajectory $\boldsymbol{u}_i$ is obtained by augmenting $\overline{\boldsymbol{u}}_i$ with the input sequence $u_i(t|k) = 0, k+N \leq t$. The infinite horizon state trajectory $\boldsymbol{x}_i$ is derived from $\overline{\boldsymbol{x}}_i$ by propagating the terminal state $x_i(k+N|k)$ using (1) and $u_i(t|k) = 0, k+N \leq t, \forall\ i = 1, 2, \ldots, M$. The cost function $\Phi_i(\cdot)$ is obtained by eliminating the state trajectory $\boldsymbol{x}_i$ from (3) using (6) and the input, state parameterization described above. The solution to the optimization problem $\mathcal{F}_i$ is denoted by $\boldsymbol{u}_i^{*(p)}$. By definition,

$$\begin{aligned}\boldsymbol{u}_i^{*(p)} &= [u_i^{*(p)}(k|k)', u_i^{*(p)}(k+1|k)', \ldots\ldots]' \text{ and} \\ \overline{\boldsymbol{u}}_i^{*(p)} &= [u_i^{*(p)}(k|k)', u_i^{*(p)}(k+1|k)', \ldots, u_i^{*(p)}(k+N-1|k)']'\end{aligned}$$

5 Terminal penalty FC-MPC

5.1 Optimization

For the quadratic form of $\phi_i(\cdot)$ given by (3), the FC-MPC optimization problem (7), for each subsystem $i = 1, 2, \ldots, M$, can be written as

$$\mathcal{F}_i \triangleq \min_{\overline{\boldsymbol{u}}_i} \quad \frac{1}{2}\overline{\boldsymbol{u}}_i{}'\mathfrak{R}_i\overline{\boldsymbol{u}}_i + \left(\boldsymbol{r}_i(x(k)) + \sum_{j\neq i}\mathbb{H}_{ij}\overline{\boldsymbol{u}}_j^{p-1}\right)'\overline{\boldsymbol{u}}_i \tag{8a}$$

subject to

$$u_i(t|k) \in \Omega_i, \quad k \leq t \leq k+N-1 \tag{8b}$$

in which

$$\begin{aligned}\mathfrak{R}_i &= \mathbb{R}_i + \sum_{j=1}^{M} E_{ji}{}'\mathbb{Q}_j E_{ji} + \sum_{j=1}^{M} E_{ji}{}'\sum_{l\neq j}\mathbb{T}_{jl}E_{li} \\ \mathbb{Q}_i &= \text{diag}\left(w_iQ_i(1), \ldots, w_iQ_i(N-1), P_{ii}\right) \\ \mathbb{T}_{ij} &= \text{diag}\left(0, \ldots, 0, P_{ij}\right) \\ \mathbb{R}_i &= \text{diag}\left(w_iR_i(0), w_iR_i(1), \ldots, w_iR_i(N-1)\right) \\ \boldsymbol{r}_i(x(k)) &= \sum_{j=1}^{M} E_{ji}{}'\, \mathbb{Q}_j\, \boldsymbol{g}_j(x(k)) + \sum_{j=1}^{M} E_{ji}{}'\sum_{l\neq j}\mathbb{T}_{jl}\boldsymbol{g}_l(x(k))\end{aligned}$$

$$\mathbb{H}_{ij} = \sum_{l=1}^{M} E_{li}{}'\mathbb{Q}_l E_{lj} + \sum_{l=1}^{M} E_{li}{}'\sum_{s\neq l}\mathbb{T}_{ls}E_{sj} \qquad\qquad \boldsymbol{g}_i(x(k)) = \sum_{j=1}^{M} f_{ij}x_j(k)$$

and

$$P = \begin{bmatrix} P_{11} & P_{12} & \ldots & \ldots & P_{1M} \\ P_{21} & P_{22} & \ldots & \ldots & P_{2M} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ P_{M1} & P_{M2} & \ldots & \ldots & P_{MM}\end{bmatrix} \tag{9}$$

is a suitable terminal penalty matrix. Restricting attention (for now) to open-loop stable systems simplifies the choice of P. For each $i = 1, 2, \ldots, M$, let $Q_i(0) = Q_i(1) = \ldots = Q_i(N-1) = Q_i$. The terminal penalty P can be obtained as the solution to the centralized Lyapunov equation

$$A'\, P\, A - P = -\mathcal{Q} \tag{10}$$

in which $\mathcal{Q} = \text{diag}(w_1Q_1, w_2Q_2, \ldots, w_MQ_M)$. The centralized Lyapunov equation (10) is solved off line. The solution to (10), P, has to be recomputed if the subsystems' models and/or cost functions are altered.

5.2 Algorithm and properties

At time k, let $p_{\max}(k)$ represent the maximum number of permissible iterates for the sampling interval. The following algorithm is employed for cooperation-based distributed MPC.

Algorithm 1 (**Terminal penalty FC-MPC**).
Given $\overline{\boldsymbol{u}}_i^0(k), x_i(k), \mathbb{Q}_i, \mathbb{R}_i,\ i = 1, 2, \ldots, M$
$p_{\max}(k) \geq 0$ and $\epsilon > 0$
$p \leftarrow 1, \rho_i \leftarrow \Gamma\epsilon, \Gamma \gg 1$
while $\rho_i > \epsilon$ for some $i = 1, 2, \ldots, M$ and $p \leq p_{\max}$
do $\forall\ i = 1, 2, \ldots, M$
$\overline{\boldsymbol{u}}_i^{*(p)} \in \arg(\mathcal{F}_i)$, (see (8))
end (do)
for each $i = 1, 2, \ldots, M$
$\overline{\boldsymbol{u}}_i^p = w_i \overline{\boldsymbol{u}}_i^{*(p)} + (1 - w_i)\, \overline{\boldsymbol{u}}_i^{p-1}$
$\rho_i = \|\overline{\boldsymbol{u}}_i^p - \overline{\boldsymbol{u}}_i^{p-1}\|$
Transmit $\overline{\boldsymbol{u}}_i^p$ to each interconnected subsystem $j = 1, 2, \ldots, M, j \neq i$.
end (for)
$p \leftarrow p + 1$
end (while)

The state trajectory for subsystem i generated by the input trajectories $\overline{\boldsymbol{u}}_1, \overline{\boldsymbol{u}}_2, \ldots, \overline{\boldsymbol{u}}_M$ and initial state z is represented as $\overline{\boldsymbol{x}}_i(\overline{\boldsymbol{u}}_1, \overline{\boldsymbol{u}}_2, \ldots, \overline{\boldsymbol{u}}_M; z)$. At each iterate p in Algorithm 1, the state trajectory for subsystem $i = 1, 2, \ldots, M$ can be calculated as $\overline{\boldsymbol{x}}_i^p(\overline{\boldsymbol{u}}_1^p, \overline{\boldsymbol{u}}_2^p, \ldots, \overline{\boldsymbol{u}}_M^p; x(k))$. At each k, $p_{\max}(k)$ represents a design limit on the number of iterates; the user may choose to terminate Algorithm 1 prior to this limit.

The infinite horizon input and state trajectories $(\boldsymbol{x}_i^p, \boldsymbol{u}_i^p)$ can be obtained following the discussion in Section 4. Denote the cooperation-based cost function after p iterates by

$$\Phi(\boldsymbol{u}_1^p, \boldsymbol{u}_2^p, \ldots, \boldsymbol{u}_M^p; x(k)) = \sum_{r=1}^{M} w_r \Phi_r \left(\boldsymbol{u}_1^p, \boldsymbol{u}_2^p, \ldots, \boldsymbol{u}_M^p; x_r(k)\right).$$

The following properties can be established for the FC-MPC formulation (8) employing Algorithm 1.

Lemma 1. *Given the distributed MPC formulation $\mathcal{F}_i$ defined in* (7) *and* (8), $\forall\ i = 1, 2, \ldots, M$, *the sequence of cost functions $\left\{\Phi(\boldsymbol{u}_1^p, \boldsymbol{u}_2^p, \ldots, \boldsymbol{u}_M^p; x(k))\right\}$ generated by Algorithm 1 is nonincreasing with iteration number p.*

A proof is given in Appendix B.

Using Lemma 1 and the fact that $\Phi(\cdot)$ is bounded below assures convergence of the sequence of cost functions with iteration number.

Consider the centralized MPC optimization problem obtained by eliminating the subsystem states using the PM equations (1), $\forall\, i = 1, 2, \ldots, M$.

$$\min_{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_M} \quad \Phi\left(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_M; x(k)\right) = \sum_{i=1}^{M} w_i \Phi_i\left(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_M; x_i(k)\right) \tag{11a}$$

subject to

$$u_i(l|k) \in \Omega_i, \quad k \leq l \leq k + N - 1, \tag{11b}$$

$$u_i(l|k) = 0, \quad k + N \leq l \tag{11c}$$

$$\forall\, i = 1, 2, \ldots, M$$

From the definition of $\phi_i(\cdot)$ (3), we have $\mathbb{R}_i > 0$. Hence, $\Re_i > 0, \forall\, i = 1, 2, \ldots, M$ (8). It follows that $\Phi_i(\cdot)$ is strictly convex. Using convexity of $\Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_M$ and strict convexity of $\Phi(\cdot)$, the solution $(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_M^*)$ to the centralized MPC optimization problem (11) exists and is unique. By definition, $\boldsymbol{u}_i^* = [\overline{\boldsymbol{u}}_i^{*\prime}, 0, 0, \ldots]$.

Lemma 2. *Consider $\Phi(\cdot)$ positive definite quadratic and $\Omega_i, \forall\, i = 1, 2, \ldots, M$ be convex, compact. Let the solution to Algorithm 1 after p iterates be $(\boldsymbol{u}_1^p, \ldots, \boldsymbol{u}_M^p)$ with an associated cost function value $\Phi(\boldsymbol{u}_1^p, \ldots, \boldsymbol{u}_M^p; x(k))$, in which $\boldsymbol{u}_i^p = [\overline{\boldsymbol{u}}_i^{p\prime}, 0, 0, \ldots]'$. Denote the unique solution to (11) by $(\boldsymbol{u}_1^*, \boldsymbol{u}_2^*, \ldots, \boldsymbol{u}_M^*)$, in which $\boldsymbol{u}_i^* = [\overline{\boldsymbol{u}}_i^{*\prime}, 0, 0, \ldots]'$, and let $\Phi(\boldsymbol{u}_1^*, \boldsymbol{u}_2^*, \ldots, \boldsymbol{u}_M^*; x(k))$ represent the optimal cost function value. The solution obtained at convergence of Algorithm 1 satisfies*

$$\lim_{p \to \infty} \Phi(\boldsymbol{u}_1^p, \boldsymbol{u}_2^p, \ldots, \boldsymbol{u}_M^p; x(k)) = \Phi(\boldsymbol{u}_1^*, \boldsymbol{u}_2^*, \ldots, \boldsymbol{u}_M^*; x(k)) \text{ and}$$

$$\lim_{p \to \infty} (\boldsymbol{u}_1^p, \boldsymbol{u}_2^p, \ldots, \boldsymbol{u}_M^p) = (\boldsymbol{u}_1^*, \boldsymbol{u}_2^*, \ldots, \boldsymbol{u}_M^*)$$

A proof is given in Appendix B.

5.3 Distributed MPC control law

At time k, let the FC-MPC algorithm (Algorithm 1) be terminated after $p(k)$ iterates, with

$$\boldsymbol{u}_i^{p(k)}(k; x(k)) = \left[u_i^{p(k)}(k; x(k))', u_i^{p(k)}(k+1; x(k))', \ldots\right]', \qquad \forall\, i = 1, 2, \ldots, M \tag{12}$$

representing the solution to Algorithm 1 after $p(k)$ cooperation-based iterates. The distributed MPC control law is obtained through a receding horizon implementation of optimal control whereby the input applied to subsystem i is

$$u_i(k) = u_i^{p(k)}(k; x(k)). \tag{13}$$

5.4 Feasibility of FC-MPC optimizations

Since $x(0) \in \mathcal{X}$, there exists a set of feasible, open-loop input trajectories $(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_M)$ such that $x_i(k) \to 0, \forall\, i = 1, 2, \ldots, M$ and k sufficiently large. Convexity of $\Omega_i, \forall\, i = 1, 2, \ldots, M$ and Algorithm 1 guarantee that given a feasible input sequence at time $k = 0$, a feasible input sequence exists for all future times. One trivial choice for a feasible input sequence at $k = 0$ is $u_i(k+l|k) = 0,\ l \geq 0,\ \forall\, i = 1, 2, \ldots, M$. This choice follows from our assumption that each Ω_i is nonempty and $0 \in \text{int}(\Omega_i)$. Existence of a feasible input sequence for each subsystem i at $k = 0$ ensures that the FC-MPC optimization problem (7), (8) has a solution for each $i = 1, 2, \ldots, M$ and all $k \geq 0$.

5.5 Initialization

At discrete time $k+1$, define $\forall\, i = 1, 2, \ldots, M$

$$\boldsymbol{u}_i^0(k+1)' = \left[u_i^{p(k)}(k+1; x(k))', u_i^{p(k)}(k+2; x(k))', \ldots, u_i^{p(k)}(k+N-1; x(k))', 0, 0, \ldots\right] \quad (14)$$

It follows that $\boldsymbol{u}_1^0(k+1), \boldsymbol{u}_2^0(k+1), \ldots, \boldsymbol{u}_M^0(k+1)$ constitute feasible subsystem input trajectories with an associated cost function $\Phi\left(\boldsymbol{u}_1^0(k+1), \boldsymbol{u}_2^0(k+1), \ldots, \boldsymbol{u}_M^0(k+1); x(k+1)\right)$.

5.6 Nominal closed-loop stability

Given the set of initial subsystem states $x_i(0),\ \forall\, i = 1, 2, \ldots, M$. Define $\widetilde{J}_N(x(0))$ to be the value of the cooperation-based cost function with the set of zero input trajectories $u_i(k+j|k) = 0,\ j \geq 0, \forall\, i = 1, 2, \ldots, M$. At time k, let $J_N^0(x(k))$ represent the value of the cooperation-based cost function with the input trajectory initialization described in (14). For notational convenience we drop the function dependence of the generated state trajectories and write $\boldsymbol{x}_i \equiv \boldsymbol{x}_i(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_M; z),\ \forall\, i = 1, 2, \ldots, M$. The value of the cooperation-based cost function after $p(k)$ iterates is denoted by $J_N^{p(k)}(x(k))$. Thus,

$$J_N^{p(k)}(x(k)) = \sum_{i=1}^{M} w_i \phi_i\left(\boldsymbol{x}_i^{p(k)}, \boldsymbol{u}_i^{p(k)}; x(k)\right) \quad (15a)$$

$$= \sum_{i=1}^{M} w_i \sum_{j=0}^{\infty} L_i\left(x_i^{p(k)}(k+j|k), u_i^{p(k)}(k+j|k)\right) \quad (15b)$$

At $k = 0$, we have using Lemma 1 that $J_N^{p(0)}(x(0)) \leq J_N^0(x(0)) = \widetilde{J}_N(x(0))$. It follows from (14) and Lemma 1 that

$$0 \leq J_N^{p(k)}(x(k)) \leq J_N^0(x(k)) = J_N^{p(k-1)}(x(k-1)) - \sum_{i=1}^{M} w_i L_i(x_i(k-1), u_i^{p(k-1)}(k-1)),\ \forall\, k > 0 \quad (16)$$

Using the above relationship recursively from time k to time 0 gives

$$J_N^{p(k)}(x(k)) \leq \widetilde{J}_N(x(0)) - \sum_{j=0}^{k-1}\sum_{i=1}^{M} w_i L_i(x_i(j), u_i^{p(j)}(j)) \leq \widetilde{J}_N(x(0)), \quad (17)$$

From (15), we have $\frac{1}{2}\lambda_{\min}(\mathcal{Q})\|x(k)\|^2 \leq J_N^{p(k)}(x(k))$. Using (17), gives $J_N^{p(k)}(x(k)) \leq \widetilde{J}_N(x(0)) = \frac{1}{2}x(0)'Px(0) \leq \frac{1}{2}\lambda_{\max}(P)\|x(0)\|^2$. From the previous two cost relationships, we obtain $\|x(k)\| \leq$

$\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(\mathcal{Q})}}\|x(0)\|$, which shows that the closed-loop system is Lyapunov stable [30, p. 265]. In fact, using the cost convergence relationship (16) the closed-loop system is also attractive, which proves asymptotic stability under the distributed MPC control law.

Lemmas 1 and 2 can be used to establish the following (stronger) exponential closed-loop stability result.

Theorem 1. *Given Algorithm 1 using the distributed MPC optimization problem* (8) *with* $N \geq 1$*. In Algorithm 1, let* $0 < p_{\max}(k) \leq p^* < \infty$, $\forall\, k \geq 0$*. If* A *is stable,* P *is obtained from* (10)*, and*

$$\begin{aligned} Q_i(0) = Q_i(1) = \ldots = Q_i(N-1) = Q_i > 0 \\ R_i(0) = R_i(1) = \cdots = R_i(N-1) = R_i > 0 \\ \forall\, i = 1, 2, \ldots, M \end{aligned}$$

then the origin is an exponentially stable equilibrium for the closed-loop system

$$x(k+1) = Ax(k) + Bu(k)$$

in which

$$u(k) = \left[u_1^{p(k)}(k; x(k))', \ldots, u_M^{p(k)}(k; x(k))'\right]'$$

for all $x(k) \in \mathbb{R}^n$ *and any* $p(k) = 1, 2, \ldots, p_{\max}(k)$*.*

A proof is given in Appendix B.

Remark 1. If $(A, \mathcal{Q}^{\frac{1}{2}})$ is detectable, then the weaker requirement $Q_i \geq 0$, $R_i > 0$, $\forall i = 1, ..., M$ is sufficient to ensure exponential stability of the closed-loop system under the distributed MPC control law.

5.7 Examples

5.7.1 Power system terminology and control area model

For the purposes of AGC, power systems are decomposed into control areas, with tie-lines providing interconnections between areas [16]. Each area typically consists of numerous generators and loads. It is common, though, for all generators in an area to be lumped as a single equivalent generator, and likewise for loads. That model is adopted in all subsequent examples. Some basic power systems terminology is provided in Table 1. The notation Δ is used to indicate a deviation from steady state. For example, $\Delta\omega$ represents a deviation in the angular frequency from its nominal operating value (60 Hz.).

Consider any control area $i = 1, 2, \ldots, M$, interconnected to control area $j, j \neq i$ through a tie line. A simplified model for such a control area i is given in (18). **Area i**

$$\frac{d\Delta\omega_i}{dt} + \frac{1}{M_i^a} D_i \Delta\omega_i + \frac{1}{M_i^a}\Delta P_{\text{tie}}^{ij} - \frac{1}{M_i^a}\Delta P_{\text{mech}_i} = -\frac{1}{M_i^a}\Delta P_{\text{L}_i} \tag{18a}$$

$$\frac{d\Delta P_{\text{mech}_i}}{dt} + \frac{1}{T_{\text{CH}_i}}\Delta P_{\text{mech}_i} - \frac{1}{T_{\text{CH}_i}}\Delta P_{\text{v}_i} = 0 \tag{18b}$$

$$\frac{d\Delta P_{\text{v}_i}}{dt} + \frac{1}{T_{\text{G}_i}}\Delta P_{\text{v}_i} - \frac{1}{T_{\text{G}_i}}\Delta P_{\text{ref}_i} + \frac{1}{R_i^f T_{\text{G}_i}}\Delta\omega_i = 0 \tag{18c}$$

Table 1: Basic power systems terminology.

ω	:	angular frequency of rotating mass
δ	:	phase angle of rotating mass
M^a	:	Angular momentum
D	:	$\frac{\text{percent change in load}}{\text{percent change in frequency}}$
P_{mech}	:	mechanical power
P_{L}	:	nonfrequency sensitive load
T_{CH}	:	charging time constant (prime mover)
P_{v}	:	steam valve position
P_{ref}	:	load reference setpoint
R^f	:	$\frac{\text{percent change in frequency}}{\text{percent change in unit output}}$
T_{G}	:	governor time constant
P_{tie}^{ij}	:	tie-line power flow between areas i and j
T_{ij}	:	tie-line (between areas i and j) stiffness coefficient
K_{ij}	:	FACTS device coefficient (regulating impedance between areas i and j)

tie-line power flow between areas i and j

$$\frac{d\Delta P_{\text{tie}}^{ij}}{dt} = T_{ij}\left(\Delta\omega_i - \Delta\omega_j\right) \tag{18d}$$

$$\Delta P_{\text{tie}}^{ji} = -\Delta P_{\text{tie}}^{ij} \tag{18e}$$

5.7.2 Performance comparison

The cumulative stage cost Λ is used as an index for comparing the performance of different MPC frameworks. Define

$$\Lambda = \frac{1}{t}\sum_{k=0}^{t-1}\sum_{i=1}^{M} L_i\left(x_i(k), u_i(k)\right). \tag{19}$$

where t is the simulation horizon.

5.7.3 Two area power system network

An example with two control areas interconnected through a tie line is considered. The controller parameters and constraints are given in Table 2. A control horizon $N = 15$ is used for each MPC. The controlled variable (CV) for area 1 is the frequency deviation $\Delta\omega_1$ and the CV for area 2 is the deviation in the tie-line power flow between the two control areas $\Delta P_{\text{tie}}^{12}$. From the control area model (18), if $\Delta\omega_1 \to 0$ and $\Delta P_{\text{tie}}^{12} \to 0$ then $\Delta\omega_2 \to 0$.

For a 25% load increase in area 2, the load disturbance rejection performance of the FC-MPC formulation is evaluated and compared against the performance of centralized MPC (cent-MPC), communication-based MPC (comm-MPC) and standard automatic generation control (AGC) with anti-reset windup. The load reference setpoint in each area is constrained between ± 0.3. In practice, a large load change, such as the one considered above, would result in curtailment of

AGC and initiation of emergency control measures such as load shedding. The purpose of this exaggerated load disturbance is to illustrate the influence of input constraints on the different control frameworks.

The relative performance of standard AGC, cent-MPC and FC-MPC (terminated after 1 iterate) rejecting the load disturbance in area 2 is depicted in Figure 2. The closed-loop trajectory of the FC-MPC controller, obtained by terminating Algorithm 1 after 1 iterate, is almost indistinguishable from the closed-loop trajectory of cent-MPC. Standard AGC performs nearly as well as cent-MPC and FC-MPC in driving the local frequency changes to zero. Under standard AGC, however, the system takes in excess of 400 seconds to drive the deviational tie-line power flow to zero. With the cent-MPC or the FC-MPC framework, the tie-line power flow disturbance is rejected in about 100 seconds. A closed-loop performance comparison of the different control frameworks is given in Table 3. The comm-MPC framework stabilizes the system but incurs a control cost that is nearly 18% greater than that incurred by FC-MPC (1 iterate). If 5 iterates per sampling interval are allowed, the performance of FC-MPC is almost identical to that of cent-MPC.

Notice from Figure 2 that the initial response of AGC is to increase generation in both areas. This causes a large deviation in the tie-line power flow. On the other hand under comm-MPC and FC-MPC, MPC-1 initially reduces area 1 generation and MPC-2 orders a large increase in area 2 generation (the area where the load disturbance occurred). This strategy enables a much more rapid restoration of tie-line power flow.

Table 2: Model parameters and input constraints for the two area power network model (Example 5.7.3).

D_1	=	2	D_2	=	2.75
R_1^f	=	0.03	R_2^f	=	0.07
M_1^a	=	3.5	M_2^a	=	4.0
T_{CH_1}	=	50	T_{CH_2}	=	10
T_{G_1}	=	40	T_{G_2}	=	25
Q_1	=	$\text{diag}(1000, 0, 0)$	Q_2	=	$\text{diag}(0, 0, 0, 1000)$
R_1	=	1	R_2	=	1
T_{12}	=	7.54	Δ_{samp}	=	0.1 sec

-0.3 $\leq \Delta P_{\text{ref}_1} \leq$ 0.3
-0.3 $\leq \Delta P_{\text{ref}_2} \leq$ 0.3

Table 3: Performance of different control formulations w.r.t. cent-MPC, $\Delta\Lambda\% = \frac{\Lambda_{\text{config}} - \Lambda_{\text{cent}}}{\Lambda_{\text{cent}}} \times 100$.

	Λ	$\Delta\Lambda\%$
standard AGC	39.26	189
comm-MPC	15.82	18.26
FC-MPC (1 iterate)	13.42	0.26
FC-MPC (5 iterates)	~ 13.38	~ 0
cent-MPC	13.38	–

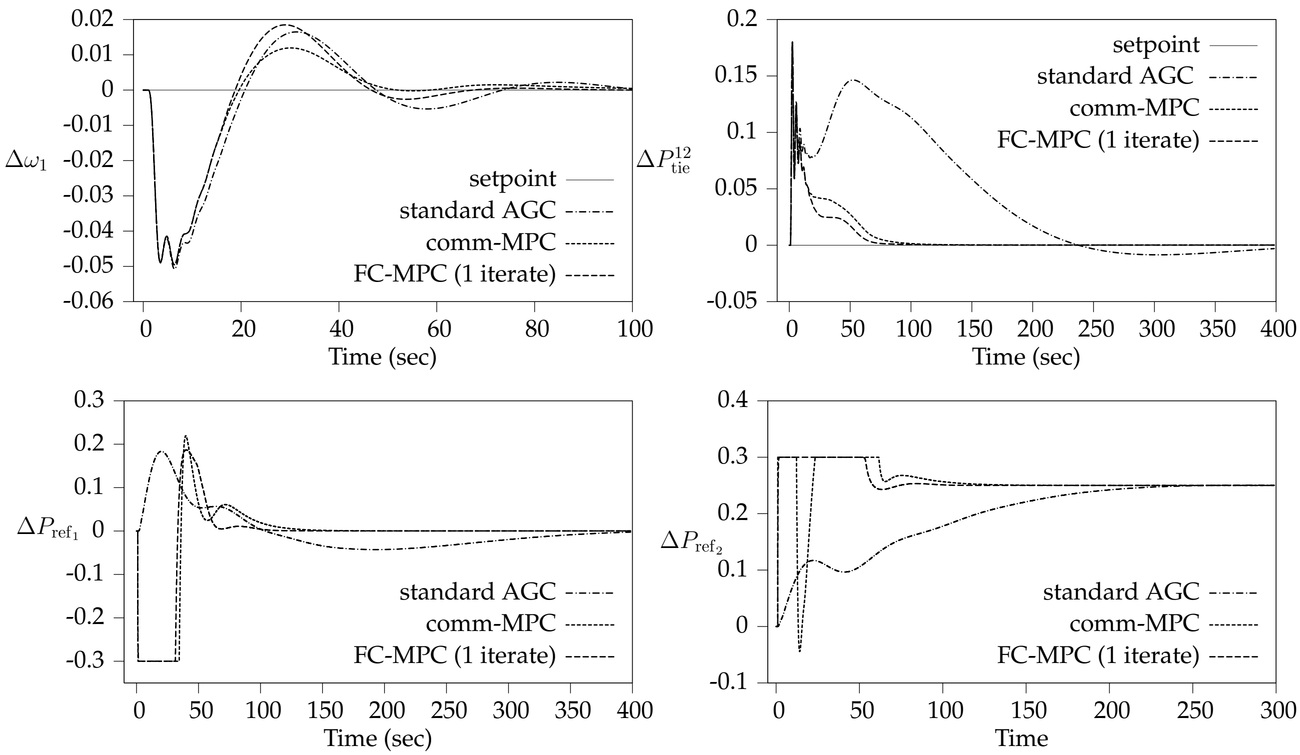

Figure 2: Performance of different control frameworks rejecting a load disturbance in area 2. Change in frequency $\Delta\omega_1$, tie-line power flow $\Delta P_{\mathrm{tie}}^{12}$ and load reference setpoints $\Delta P_{\mathrm{ref}_1}, \Delta P_{\mathrm{ref}_2}$.

5.7.4 Four area power system network

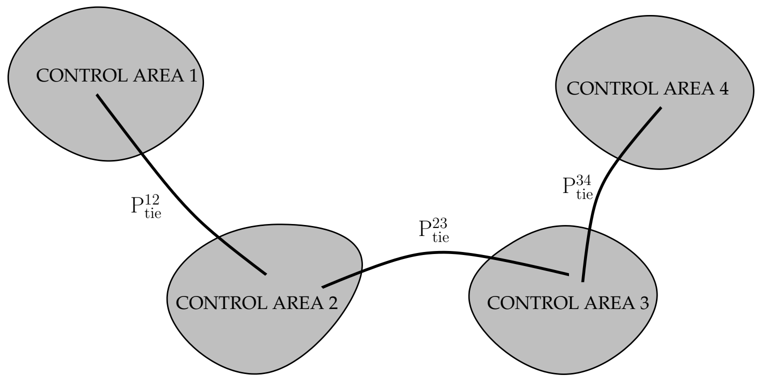

Figure 3: Four area power system.

Consider the four area power system shown in Figure 3. The model for each control area follows from (18). Model parameters are given in Table 4. In each control area, a change in local power demand (load) alters the nominal operating frequency. The MPC in each control area i manipulates the load reference setpoint P_{ref_i} to drive the frequency deviations $\Delta\omega_i$ and tie-line power flow deviations $\Delta P^{ij}_{\text{tie}}$ to zero. Power flow through the tie lines gives rise to interactions among the control areas. Hence a load change in area 1, for instance, causes a transient frequency change in all control areas.

The relative performance of cent-MPC, comm-MPC and FC-MPC is analyzed for a 25% load increase in area 2 and a simultaneous 25% load drop in area 3. This load disturbance occurs at 5 sec. For each MPC, we choose a prediction horizon of $N = 20$. In the comm-MPC and FC-MPC formulations, the load reference setpoint $(\Delta P_{\text{ref}_i})$ in each area is manipulated to reject the load disturbance and drive the change in local frequencies $(\Delta\omega_i)$ and tie-line power flows $(\Delta P^{ij}_{\text{tie}})$ to zero. In the cent-MPC framework, a single MPC manipulates all four ΔP_{ref_i}. The load reference setpoint for each area is constrained between ± 0.5.

The performance of cent-MPC, comm-MPC and FC-MPC (1 iterate) are shown in Figure 4. Only $\Delta\omega_2$ and $\Delta P^{23}_{\text{tie}}$ are shown as the frequency and tie-line power flow deviations in the other areas display similar qualitative behavior. Likewise, only ΔP_{ref_2} and ΔP_{ref_3} are shown as other load reference setpoints behave similarly. The control costs are given in Table 5. Under comm-MPC, the load reference setpoints for areas 2 and 3 switch repeatedly between their upper and lower saturation limits. Consequently, the power system network is unstable under comm-MPC. The closed-loop performance of the FC-MPC formulation, terminated after just 1 iterate, is within 26% of cent-MPC performance. If the FC-MPC algorithm is terminated after 5 iterates, the performance of FC-MPC is within 4% of cent-MPC performance. By allowing the cooperation-based iterative process to converge, the closed-loop performance of FC-MPC can be driven to within any pre-specified tolerance of cent-MPC performance.

5.7.5 Two area power system with FACTS device

In this example, we revisit the two area network considered in Section 5.7.3. In this case though, a FACTS device is employed by area 1 to manipulate the effective impedance of the tie line and

Table 4: Model, regulator parameters and input constraints for four area power network of Figure 3.

D_1 = 3	D_2 =0.275
R_1^f =0.03	R_2^f = 0.07
M_1^a = 4	M_2^a = 40
T_{CH_1}= 5	T_{CH_2} = 10
T_{G_1} = 4	T_{G_2} = 25
D_3 = 2.0	D_4 = 2.75
R_3^f =0.04	R_4^f = 0.03
M_3^a = 35	M_4^a = 10
T_{CH_3}= 20	T_{CH_4} = 10
T_{G_3} = 15	T_{G_4} = 5
T_{12} =2.54	T_{23} = 1.5
T_{34} = 2.5	Δ_{samp} = 1 sec

-0.5$\leq \Delta P_{\mathrm{ref}_1} \leq$0.5
-0.5$\leq \Delta P_{\mathrm{ref}_2} \leq$0.5
-0.5$\leq \Delta P_{\mathrm{ref}_3} \leq$0.5
-0.5$\leq \Delta P_{\mathrm{ref}_4} \leq$0.5

$Q_1 = \mathrm{diag}(5, 0, 0)$	$R_1 = 1$
$Q_2 = \mathrm{diag}(5, 0, 0, 5)$	$R_2 = 1$
$Q_3 = \mathrm{diag}(5, 0, 0, 5)$	$R_3 = 1$
$Q_4 = \mathrm{diag}(5, 0, 0, 5)$	$R_4 = 1$

Area	States	MVs	CVs
1	$\Delta\omega_1, \Delta P_{\mathrm{mech}_1}, \Delta P_{v_1}$	$\Delta P_{\mathrm{ref}_1}$	$\Delta\omega_1$
2	$\Delta\omega_2, \Delta P_{\mathrm{mech}_2}, \Delta P_{v_2}, \Delta P_{\mathrm{tie}}^{12}$	$\Delta P_{\mathrm{ref}_2}$	$\Delta\omega_2, \Delta P_{\mathrm{tie}}^{12}$
3	$\Delta\omega_3, \Delta P_{\mathrm{mech}_3}, \Delta P_{v_3}, \Delta P_{\mathrm{tie}}^{23}$	$\Delta P_{\mathrm{ref}_3}$	$\Delta\omega_3, \Delta P_{\mathrm{tie}}^{23}$
4	$\Delta\omega_4, \Delta P_{\mathrm{mech}_4}, \Delta P_{v_4}, \Delta P_{\mathrm{tie}}^{34}$	$\Delta P_{\mathrm{ref}_4}$	$\Delta\omega_4, \Delta P_{\mathrm{tie}}^{34}$

Table 5: Performance of different MPC frameworks relative to cent-MPC, $\Delta\Lambda\% = \frac{\Lambda_{\mathrm{config}} - \Lambda_{\mathrm{cent}}}{\Lambda_{\mathrm{cent}}} \times 100$.

	$\Lambda \times 10^{-2}$	$\Delta\Lambda\%$
cent-MPC	7.6	–
comm-MPC	$\uparrow \infty$	$\uparrow \infty$
FC-MPC (1 iterate)	9.6	26
FC-MPC (5 iterates)	7.87	3.7

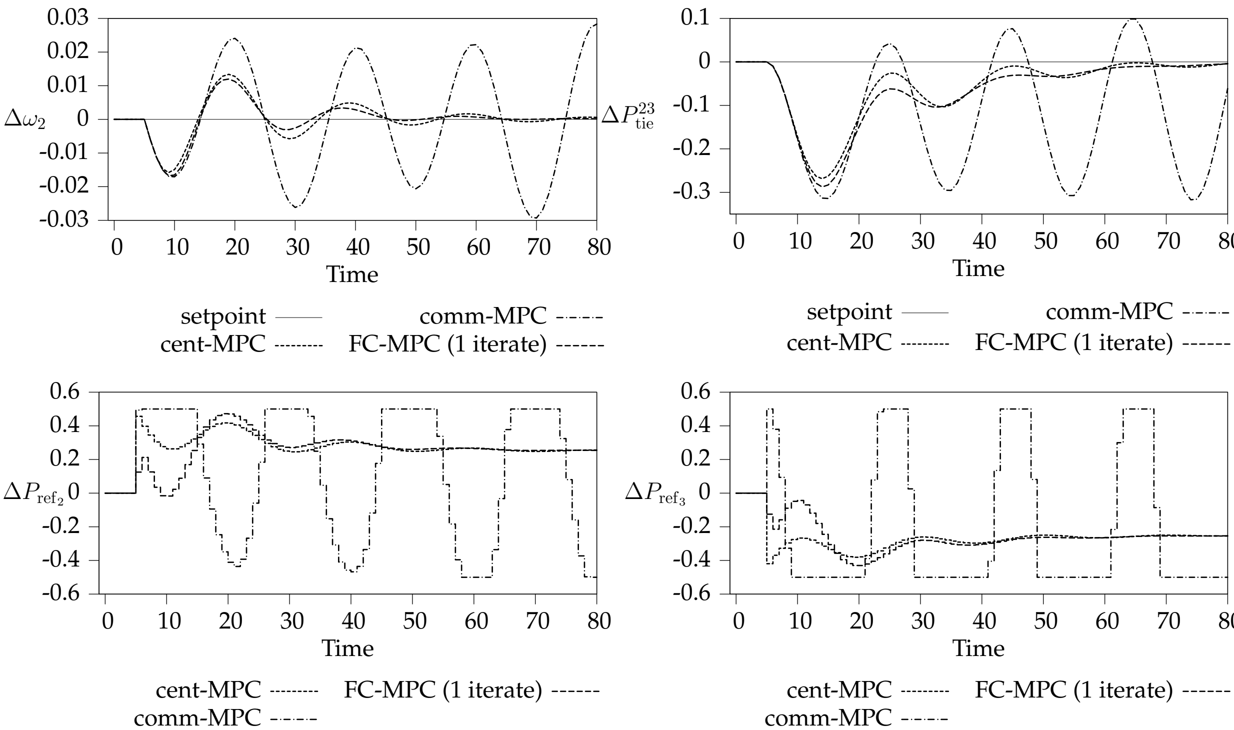

Figure 4: Performance of different control frameworks rejecting a load disturbance in areas 2 and 3. Change in frequency $\Delta\omega_2$, tie-line power flow $\Delta P_{\text{tie}}^{23}$ and load reference setpoints $\Delta P_{\text{ref}_2}, \Delta P_{\text{ref}_3}$.

control power flow between the two interconnected control areas. The control area model follows from (18). In order to incorporate the FACTS device, (18a) in area 1 is replaced by

$$\frac{d\Delta\delta_{12}}{dt} = (\Delta\omega_1 - \Delta\omega_2)$$
$$\frac{d\Delta\omega_1}{dt} = -\frac{1}{M_1^a}D_1\Delta\omega_1 - \frac{1}{M_1^a}T_{12}\Delta\delta_{12} + \frac{1}{M_1^a}K_{12}\Delta X_{12} + \frac{1}{M_1^a}\Delta P_{\text{mech}_1} - \frac{1}{M_1^a}\Delta P_{\text{L}_1}$$

and in area 2 by

$$\frac{d\Delta\omega_2}{dt} = -\frac{1}{M_2^a}D_2\Delta\omega_2 + \frac{1}{M_2^a}T_{12}\Delta\delta_{12} - \frac{1}{M_2^a}K_{12}\Delta X_{12} + \frac{1}{M_2^a}\Delta P_{\text{mech}_2} - \frac{1}{M_2^a}\Delta P_{\text{L}_2}$$

where ΔX_{12} is the impedence deviation induced by the FACTS device. The tie-line power flow deviation becomes

$$\Delta P_{\text{tie}}^{12} = -\Delta P_{\text{tie}}^{21} = T_{12}\Delta\delta_{12} - K_{12}\Delta X_{12}$$

Notice that if $\Delta X_{12} = 0$, the model reverts to (18). Controller parameters and constraints are given in Table 6. The MPC for area 1 manipulates ΔP_{ref_1} and ΔX_{12} to drive $\Delta\omega_1$ and the relative phase difference $\Delta\delta_{12} = \Delta\delta_1 - \Delta\delta_2$ to zero. The MPC for area 2 manipulates ΔP_{ref_2} to drive $\Delta\omega_2$ to zero.

Table 6: Model parameters and input constraints for the two area power network model. FACTS device operated by area 1.

D_1 =	3	D_2 =	0.275
R_1^f =	0.03	R_2^f =	0.07
M_1^a =	4	M_2^a =	40
T_{CH_1} =	5	T_{CH_2} =	10
T_{G_1} =	4	T_{G_2} =	25
T_{12} =	2.54	K_{12} =	1.95
Q_1 =	$\text{diag}(100, 0, 0, 100)$	Q_2 =	$\text{diag}(100, 0, 0)$
R_1 =	1	R_2 =	1
N =	15	Δ_{samp}=	1 sec

-0.3 $\leq \Delta P_{\text{ref}_1} \leq$ 0.3
-0.1 $\leq \Delta X_{12} \leq$ 0.1
-0.3 $\leq \Delta P_{\text{ref}_2} \leq$ 0.3

The relative performance of cent-MPC, comm-MPC and FC-MPC rejecting a simultaneous 25% increase in the load of areas 1 and 2 is investigated. The closed-loop performance of the different MPC frameworks is shown in Figure 5. The associated control costs are given in Table 7. The performance of FC-MPC (1 iterate) is within 28% of cent-MPC performance. The performance of comm-MPC, on the other hand, is highly oscillatory and significantly worse than that of FC-MPC (1 iterate). While comm-MPC is stabilizing, the system takes nearly 400 sec to reject the load disturbance. With FC-MPC (1 iterate), the load disturbance is rejected in less than 80 sec. If 5 iterates per sampling interval are possible, the FC-MPC framework achieves performance that is within 2.5% of cent-MPC performance.

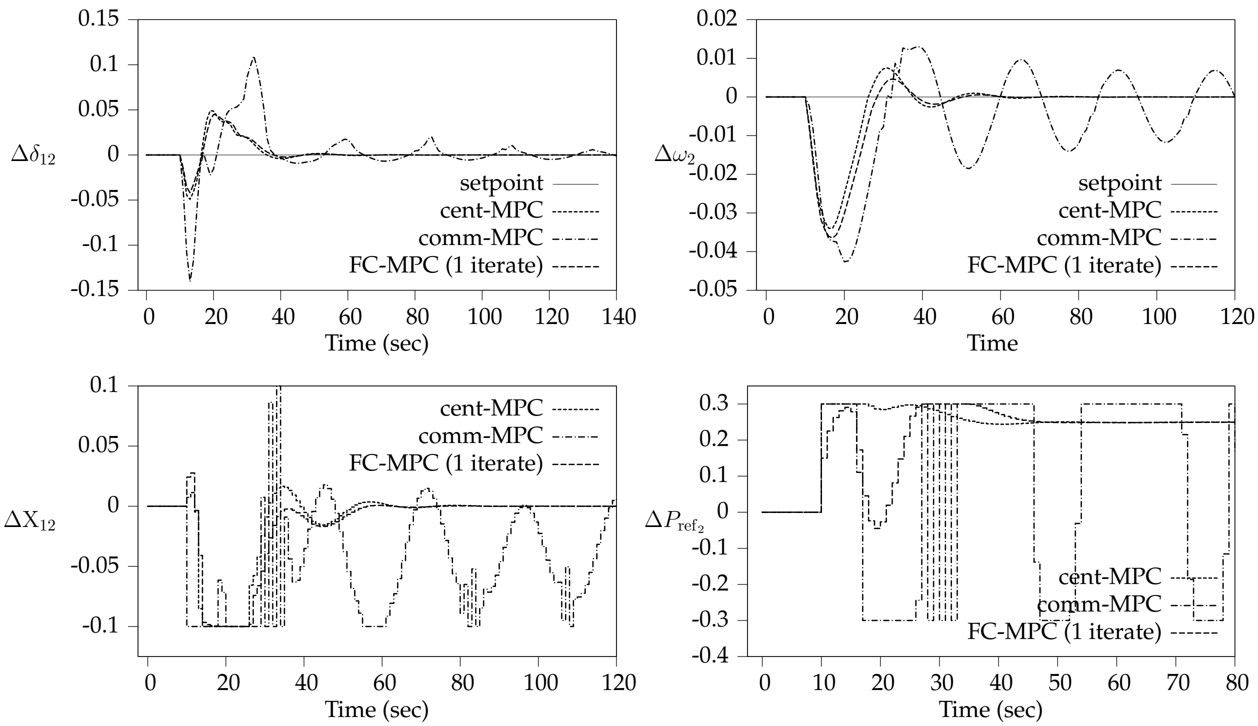

Figure 5: Performance of different control frameworks rejecting a load disturbance in area 2. Change in relative phase difference $\Delta\delta_{12}$, frequency $\Delta\omega_2$, tie-line impedence ΔX_{12} due to the FACTS device and load reference setpoint ΔP_{ref_2}.

Table 7: Performance of different MPC frameworks relative to cent-MPC, $\Delta\Lambda\% = \frac{\Lambda_{\text{config}} - \Lambda_{\text{cent}}}{\Lambda_{\text{cent}}} \times 100$.

	$\Lambda \times 10^{-2}$	$\Delta\Lambda\%$
cent-MPC	3.06	–
comm-MPC	9.53	211
FC-MPC (1 iterate)	3.92	28
FC-MPC (5 iterates)	3.13	2.3

6 Extensions

6.1 Penalty and constraints on the rate of change of input

The proposed distributed MPC framework can be extended to penalize and constrain the rate of change of inputs. The notation $\Delta u_i(k) = u_i(k) - u_i(k-1)$ represents the change in input from time $k-1$ to time k. The constraints on the rate of change of input are of the form $\Delta u_i^{\min} \leq \Delta u_i \leq \Delta u_i^{\max}, i = 1, 2, \ldots, M$, which represent limits on how fast the actuators/valves can move in reality. The stage cost $L_i(x_i(k), u_i(k), \Delta u_i(k))$ for each subsystem $i = 1, 2, \ldots, M$ at time k is defined as

$$L_i(x_i(k), u_i(k), \Delta u_i(k) = \frac{1}{2}\left[x_i(k)'Q_i x_i(k) + u_i(k)'R_i u_i(k) + \Delta u_i(k)'S_i \Delta u_i(k)\right] \tag{20}$$

in which $Q_i, R_i, S_i \geq 0, R_i + S_i > 0$ with $(A_i, Q_i^{1/2})$ detectable. To convert (20) to standard form (2), we augment the state vector for subsystem i with the subsystem input $u_i(k-1)$ obtained at time $k-1$ [22]. The stage cost (20) can be re-written as

$$L_i(z_i(k), u_i(k)) = \frac{1}{2}\left[z_i(k)'\widetilde{Q}_i z_i(k) + u_i(k)'\widetilde{R}_i u_i(k) + 2z_i(k)'\widetilde{M}_i u_i(k)\right] \tag{21}$$

in which

$$z_i(k) = \begin{bmatrix} x_i(k) \\ u_i(k-1) \end{bmatrix} \qquad \widetilde{Q}_i = \begin{bmatrix} Q_i & \\ & S_i \end{bmatrix}$$

$$\widetilde{R}_i = R_i + S_i \qquad \widetilde{M}_i = \begin{bmatrix} 0 \\ -S_i \end{bmatrix}$$

The augmented PM for subsystem $i = 1, 2, \ldots, M$ is

$$z_i(k+1) = \widetilde{A}_{ii} z_i(k) + \widetilde{B}_{ii} u_i(k) + \sum_{j \neq i} \left[\widetilde{A}_{ij} z_j(k) + \widetilde{B}_{ij} u_j(k)\right] \tag{22}$$

in which

$$\widetilde{A}_{ij} = \begin{bmatrix} A_{ij} & 0 \\ 0 & 0 \end{bmatrix}, \; \forall\, i, j = 1, 2, \ldots, M$$

$$\widetilde{B}_{ii} = \begin{bmatrix} B_{ii} \\ I \end{bmatrix}, \quad \widetilde{B}_{ij} = \begin{bmatrix} B_{ij} \\ 0 \end{bmatrix}, \; \forall\, i, j = 1, 2, \ldots, M, j \neq i$$

The cost function for subsystem i is defined as

$$\phi_i(\boldsymbol{z}_i, \boldsymbol{u}_i; x(k)) = \sum_{j=k}^{\infty} L_i(z_i(j|k), u_i(j|k)) \tag{23}$$

The constraints on the rate of change of input for each subsystem $i = 1, 2, \ldots, M$ can, therefore, be expressed as

$$\overline{\boldsymbol{\Delta}\mathbf{u}}_i^{\min} \leq \mathcal{D}_i\, \overline{\boldsymbol{u}}_i \leq \overline{\boldsymbol{\Delta}\mathbf{u}}_i^{\max} \tag{24a}$$

in which

$$\overline{\mathbf{\Delta u}}_i^{\min} = \begin{bmatrix} \Delta u_i^{\min} - u_i(k-1) \\ \Delta u_i^{\min} \\ \vdots \\ \Delta u_i^{\min} \end{bmatrix} \qquad \overline{\mathbf{\Delta u}}_i^{\max} = \begin{bmatrix} \Delta u_i^{\max} - u_i(k-1) \\ \Delta u_i^{\max} \\ \vdots \\ \Delta u_i^{\max} \end{bmatrix} \tag{24b}$$

$$\mathcal{D}_i = \begin{bmatrix} I & & & & \\ -I & I & & & \\ & -I & I & & \\ & & & \ddots & \\ & & & -I & I \end{bmatrix} \tag{24c}$$

Following the model manipulation described in Appendix A, with each (A_{ij}, B_{ij}) pair replaced by the corresponding $(\widetilde{A}_{ij}, \widetilde{B}_{ij})$ pair (from the augmented PM (22)), gives

$$\overline{\boldsymbol{z}}_i = E_{ii}\overline{\boldsymbol{u}}_i + f_{ii}z_i(k) + \sum_{j \neq i}[E_{ij}\overline{\boldsymbol{u}}_j + f_{ij}z_j(k)], \quad \forall\, i = 1, 2, \ldots, M \tag{25}$$

in which $\overline{\boldsymbol{z}}_i = [z_i(k+1|k)', \ldots, z_i(k+N|k)']'$. Similar to Section 4, the augmented state vector z_i in (23) can be eliminated using (25). The cost function $\phi_i(\cdot)$ can therefore be re-written as a function $\Phi_i(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_M; z(k))$ where $z = [z_1', z_2', \ldots, z_M']'$. For $\phi_i(\cdot)$ defined in (23), the FC-MPC optimization problem for subsystem i is

$$\overline{\boldsymbol{u}}_i^{*(p)} \in \arg\min_{\overline{\boldsymbol{u}}_i} \quad \frac{1}{2}\overline{\boldsymbol{u}}_i'\mathfrak{R}_i\overline{\boldsymbol{u}}_i + \left(\boldsymbol{r}_i(z(k)) + \sum_{j \neq i}^{M} \mathbb{H}_{ij}\overline{\boldsymbol{u}}_j^{p-1}\right)' \overline{\boldsymbol{u}}_i \tag{26a}$$

subject to

$$u_i(j|k) \in \Omega_i, \quad k \leq j \leq k+N-1 \tag{26b}$$

$$\overline{\mathbf{\Delta u}}_i^{\min} \leq \mathcal{D}_i\, \overline{\boldsymbol{u}}_i \leq \overline{\mathbf{\Delta u}}_i^{\max} \tag{26c}$$

in which

$$\mathfrak{R}_i = \left(\mathbb{R}_i + E_{ii}'\mathbb{Q}_i E_{ii} + 2E_{ii}'\mathbb{M}_i\right) + \sum_{j\neq i}^{M} E_{ji}'\mathbb{Q}_j E_{ji} + \sum_{j=1}^{M} E_{ji}' \sum_{l\neq j} \mathbb{T}_{jl} E_{li}$$

$$\mathbb{H}_{ij} = \sum_{l=1}^{M} E_{li}'\mathbb{Q}_l E_{lj} + \mathbb{M}_i' E_{ij} + E_{ji}'\mathbb{M}_j + \sum_{l=1}^{M} E_{li}' \sum_{s\neq l} \mathbb{T}_{ls} E_{sj}$$

$$\boldsymbol{r}_i(z(k)) = \left(E_{ii}'\mathbb{Q}_i \boldsymbol{g}_i(z(k)) + \mathbb{M}_i'\boldsymbol{g}_i(z(k)) + \boldsymbol{p}_i z_i(k)\right) + \sum_{j\neq i}^{M} E_{ji}'\mathbb{Q}_j \boldsymbol{g}_j(z(k)) + \sum_{j=1}^{M} E_{ji}' \sum_{l\neq j} \mathbb{T}_{jl} \boldsymbol{g}_l(z(k))$$

$$\mathbb{Q}_i = \text{diag}\left(w_i\widetilde{Q}_i(1), \ldots, w_i\widetilde{Q}_i(N-1), \widetilde{P}_{ii}\right)$$

$$\mathbb{T}_{ij} = \text{diag}\left(0, \ldots, 0, \widetilde{P}_{ij}\right)$$

$$\mathbb{R}_i = \text{diag}\left(w_i\widetilde{R}_i(0), w_i\widetilde{R}_i(1), \ldots, w_i\widetilde{R}_i(N-1)\right)$$

$$\boldsymbol{p}_i' = \begin{bmatrix} w_i\widetilde{M}_i & 0 & \ldots & 0 \end{bmatrix}$$

$$\mathbb{M}_i = \begin{pmatrix} 0 & w_i\widetilde{M}_i & & & \\ & 0 & w_i\widetilde{M}_i & & \\ & & 0 & \ddots & \\ & & & \ddots & w_i\widetilde{M}_i \\ 0 & 0 & \ldots & \ldots & 0 \end{pmatrix}$$

The terminal penalty $\widetilde{P}$ is obtained as the solution to the centralized Lyapunov equation (10) with each A_{ij}, Q_i replaced by $\widetilde{A}_{ij}, \widetilde{Q}_i$ respectively $\forall\, i, j = 1, 2, \ldots, M$.

6.2 Unstable systems

In the development of the proposed distributed MPC framework, it was convenient to assume that the system is open-loop stable. That assumption can be relaxed however. For any real matrix $A \in \mathbb{R}^{n\times n}$ the **Schur decomposition** [13, p. 341] is defined as

$$A = \begin{bmatrix} U_s & U_u \end{bmatrix} \begin{bmatrix} A_s & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} U_s' \\ U_u' \end{bmatrix} \tag{27}$$

in which $U = \begin{bmatrix} U_s & U_u \end{bmatrix}$ is a real and orthogonal $n \times n$ matrix, the eigenvalues of A_s are strictly inside the unit circle, and the eigenvalues of A_u are on or outside the unit circle. Let $U_u' = [U_{u_1}', U_{u_2}', \ldots, U_{u_M}']$.

Define $\mathcal{T}_i' = [0, 0, \ldots, I]$ such that $x_i(k+N|k) = \mathcal{T}_i'\overline{\boldsymbol{x}}_i(k)$. To ensure closed-loop stability while dealing with open-loop unstable systems, a terminal state constraint that forces the unstable modes to be at the origin at the end of the control horizon is necessary. The control horizon must satisfy $N \geq r$, in which r is the number of unstable modes.

For each subsystem $i = 1, 2, \ldots, M$ at time k, the terminal state constraint can be written as

$$\begin{aligned} U_u' x(k+N|k) &= \sum_i U_{u_i}' x_i(k+N|k) \\ &= \sum_i (\mathcal{T}_i U_{u_i})'\overline{\boldsymbol{x}}_i(k) \\ &= 0 \end{aligned} \tag{28}$$

From (28) and (6), the terminal state constraint can be re-written as a coupled input constraint of the form

$$\mathcal{J}_1\overline{\boldsymbol{u}}_1 + \mathcal{J}_2\overline{\boldsymbol{u}}_2 + \ldots + \mathcal{J}_M\overline{\boldsymbol{u}}_M = -\mathbf{c}(x(k)) \tag{29a}$$

in which

$$\mathcal{J}_i = \sum_{j=1}^{M} \left(\mathcal{T}_j U_{u_j}\right)' E_{ji} \qquad \mathbf{c}(x(k)) = \sum_{j=1}^{M} \left(\mathcal{T}_j U_{u_j}\right)' \boldsymbol{g}_j(x(k)) \tag{29b}$$
$$\forall\, i = 1, 2, \ldots, M$$

Using the definitions in (8), the FC-MPC optimization problem for each $i = 1, 2, \ldots, M$ is

$$\mathcal{F}_i^{\text{unstb}} \triangleq \min_{\overline{\boldsymbol{u}}_i} \quad \frac{1}{2}\overline{\boldsymbol{u}}_i{}'\mathfrak{R}_i\overline{\boldsymbol{u}}_i + \left(\boldsymbol{r}_i(x(k)) + \sum_{j\neq i}\mathbb{H}_{ij}\overline{\boldsymbol{u}}_j^{p-1}\right)'\overline{\boldsymbol{u}}_i \tag{30a}$$

subject to

$$u_i(t|k) \in \Omega_i, \quad k \leq t \leq k + N - 1 \tag{30b}$$

$$\mathcal{J}_i\overline{\boldsymbol{u}}_i + \sum_{j\neq i}^{M}\mathcal{J}_j\overline{\boldsymbol{u}}_j^{p-1} = -\mathbf{c}(x(k)) \tag{30c}$$

The optimization problem (30) is solved within the framework of Algorithm 1. To initialize Algorithm 1, a simple linear/quadratic program is solved to compute subsystem input trajectories that satisfy the constraints in (30) for each subsystem. To ensure feasibility of the end constraint (30c), it is assumed that the initial state $x(0) \in \mathcal{X}_N$, the N-step stabilizable set for the system. Since $\mathcal{X}_N \subseteq \mathcal{X}$, the system is constrained stabilizable. It follows from Algorithm 1, Section 5.3 and Section 5.5 that $\mathcal{X}_N$ is an invariant set for the nominal closed-loop system, which ensures that the optimization problem (30) is feasible for each subsystem $i = 1, 2, \ldots, M$ for all $k \geq 0$ and any $p(k) > 0$. It can be shown that all iterates generated by Algorithm 1 are systemwide feasible, the cooperation-based cost function $\Phi(\boldsymbol{u}_1^p, \boldsymbol{u}_2^p, \ldots, \boldsymbol{u}_M^p; x(k))$ is a nonincreasing function of the iteration number p, and the sequence of cooperation-based iterates is convergent [3]. An important distinction, which arises due to the presence of the coupled input constraint (30c), is that the limit points of Algorithm 1 (now solving optimization problem (30) instead) are no longer necessarily optimal. The distributed MPC control law based on any intermediate iterate is feasible and closed-loop stable, but may not achieve optimal (centralized) performance at convergence of the iterates.

7 Terminal control FC-MPC

The terminal penalty-based FC-MPC framework considered earlier utilizes a suboptimal parameterization of the postulated input trajectories, and at convergence achieves performance that is within a pre-specified tolerance of a modified infinite horizon optimal control problem (11) . The

[3] The proof is identical to that presented for Lemma 1 and is, therefore, omitted.

performance of terminal penalty FC-MPC at convergence is infinite horizon optimal (solution of (4)) only in the limit as $N \to \infty$. The motivation behind terminal control-based FC-MPC is to achieve infinite horizon optimal (centralized, constrained LQR) performance at convergence using finite values of N.

The standard notation $\mathcal{O}_\infty$ is used to represent the maximal output admissible set [12] for the centralized system (A, B, C). Since $\Omega_i, \forall\, i = 1, 2, \ldots, M$ is convex, Ω is convex. Hence, from [12, Theorem 2,1], $\mathcal{O}_\infty$ is convex. We will assume that each Ω_i is a polytope *i.e.*,

$$\Omega_i \triangleq \left\{ u_i \,\middle|\, D_i u_i \leq d_i, d_i > 0 \right\} \tag{31}$$

The determination of $\mathcal{O}_\infty$, in this case, involves the solution to a set of linear programs.

Let K denote the optimal, centralized linear quadratic regulator (LQR) gain and let Π denote the solution to the corresponding centralized discrete steady-state Riccati equation *i.e.*,

$$\Pi = \mathcal{Q} + A'\Pi A - A'\Pi B(\mathcal{R} + B'\Pi B)^{-1} B'\Pi A \tag{32a}$$

$$K = -(\mathcal{R} + B'\Pi B)^{-1} B'\Pi A \tag{32b}$$

in which $\mathcal{Q} = \mathrm{diag}(w_1 Q_1, w_2 Q_2, \ldots, w_M Q_M)$ and $\mathcal{R} = \mathrm{diag}(w_1 R_1, w_2 R_2, \ldots, w_M R_M)$. Conditions for existence of a solution to (32) are well known [3, 7].

Using a subsystem-wise partitioning for K and Π gives

$$K = \begin{bmatrix} K_{11} & K_{12} & \ldots & K_{1M} \\ K_{21} & K_{22} & \ldots & K_{2M} \\ \vdots & \ddots & \ddots & \vdots \\ K_{M1} & K_{M2} & \ldots & K_{MM} \end{bmatrix} \qquad \Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \ldots & \Pi_{1M} \\ \Pi_{21} & \Pi_{22} & \ldots & \Pi_{2M} \\ \vdots & \ddots & \ddots & \vdots \\ \Pi_{M1} & \Pi_{M2} & \ldots & \Pi_{MM} \end{bmatrix} \tag{33}$$

7.1 Optimization

The terminal control FC-MPC optimization problem for subsystem i is

$$\min_{\boldsymbol{u}_i} \quad \sum_{r=1}^{M} w_r \Phi_r \left(\boldsymbol{u}_1^{p-1}, \ldots, \boldsymbol{u}_{i-1}^{p-1}, \boldsymbol{u}_i, \boldsymbol{u}_{i+1}^{p-1}, \ldots, \boldsymbol{u}_M^{p-1}; x_r(k) \right) \tag{34a}$$

subject to

$$u_i(t|k) \in \Omega_i, \;\; k \leq t \leq k + N - 1 \tag{34b}$$

$$u_i(t|k) = K_{ii} x_i(t|k) + \sum_{j \neq i} K_{ij} x_j(t|k), \;\; k + N \leq t \tag{34c}$$

To explicitly consider, for each subsystem, the dependence of the finite input trajectory on the control horizon length (N), we write

$$\overline{\boldsymbol{u}}_i(k; N) = \left[u_i(k|k)', u_i(k+1|k)', \ldots, u_i(k+N-1|k)' \right]', \; \forall\, i = 1, 2, \ldots, M \tag{35}$$

The finite state trajectory for subsystem i, generated by the set of input trajectories $\overline{\boldsymbol{u}}_1(k; N), \ldots,$ $\overline{\boldsymbol{u}}_M(k; N)$ is represented as $\overline{\boldsymbol{x}}_i(\overline{\boldsymbol{u}}_1(k; N), \ldots, \overline{\boldsymbol{u}}_M(k; N); x(k))$. For notational convenience, we write $\overline{\boldsymbol{x}}_i(k; N) \leftarrow \overline{\boldsymbol{x}}_i(\overline{\boldsymbol{u}}_1(k; N), \ldots, \overline{\boldsymbol{u}}_M(k; N); x(k))$.

Using the definitions in (8) and redefining

$$\mathbb{Q}_i = \text{diag}\left(w_i Q_i(1), \ldots, w_i Q_i(N-1), \Pi_{ii}\right) \qquad \mathbb{T}_{ij} = \text{diag}\left(0, \ldots, 0, \Pi_{ij}\right)$$

the optimization problem (34), for $\phi_i(\cdot)$ quadratic (3), can be written as

$$\mathcal{P}_i^N \triangleq \min_{\overline{\boldsymbol{u}}_i(k;N)} \quad \frac{1}{2}\overline{\boldsymbol{u}}_i(k;N)'\mathfrak{R}_i\overline{\boldsymbol{u}}_i(k;N) + \left(\boldsymbol{r}_i(x(k)) + \sum_{j\neq i}\mathbb{H}_{ij}\overline{\boldsymbol{u}}_j^{p-1}(k;N)\right)' \overline{\boldsymbol{u}}_i(k;N) \tag{36a}$$

subject to

$$u_i(t|k) \in \Omega_i, \quad k \leq t \leq k+N-1 \tag{36b}$$

Remark 2. At each iterate p, the validity of the terminal set constraint $x^p(k+N|k) \in \mathcal{O}_\infty$ must be verified. A subsystem-based procedure to certify the above condition (*i.e.*, ensure feasibility of the unconstrained centralized control law) is presented in Section 7.3.

Remark 3. Consider a set of input trajectories $\overline{\boldsymbol{u}}_1(k;N), \ldots, \overline{\boldsymbol{u}}_M(k;N)$ such that $x(k+N|k) \in \mathcal{O}_\infty$. Since $\mathcal{O}_\infty$ is a positively invariant set for the system $x(k+1) = (A+BK)x(k)$, an infinite input trajectory, $\boldsymbol{u}_i(k)$, can be constructed as

$$\boldsymbol{u}_i(k) = \left[\overline{\boldsymbol{u}}_i(k;N)', u_i(k+N|k)', u_i(k+N+1|k)', \ldots\ldots\right], \ \forall\, i = 1, 2, \ldots, M \tag{37}$$

in which $u_i(k+N+j|k) = \mathcal{V}_i{}'\left[K(A+BK)^j x(k+N|k)\right]$, $0 \leq j$. The matrix $\mathcal{V}_i = [0, \ldots, I_{m_i}, \ldots]'$ is defined such that $u_i = \mathcal{V}_i{}'u$.

7.2 Initialization

To initialize the terminal control FC-MPC algorithm, it is necessary to calculate a set of subsystem input trajectories that steers the system state to $\mathcal{O}_\infty$ at the end of each MPC's control horizon. For the initial system state $x(0) \notin \mathcal{O}_\infty$, such a set of subsystem input trajectories can be computed by solving a simple quadratic program (QP). One formulation for this initialization QP is described as follows,

$$\mathcal{L}^{\mathrm{N}}(x(k)) = \arg \min_{\widetilde{\boldsymbol{u}}(k;N)} \|\widetilde{\boldsymbol{u}}(k;N)\|^2 \tag{38a}$$

subject to

$$\mathcal{T}'\left(\mathbb{E}\,\widetilde{\boldsymbol{u}}(k;N) + \boldsymbol{g}(x(k))\right) \in \mathcal{O}_\infty \tag{38b}$$

$$u_i(k+j|k) \in \Omega_i, \qquad \begin{array}{l} j = 0, 1, \ldots, N-1 \\ i = 1, 2, \ldots, M \end{array} \tag{38c}$$

in which

$$\mathbb{E} = \begin{bmatrix} E_{11} & E_{12} & \dots & E_{1M} \\ E_{21} & E_{22} & \dots & E_{2M} \\ \vdots & \ddots & \ddots & \vdots \\ E_{1M} & E_{2M} & \dots & E_{MM} \end{bmatrix} \qquad \mathcal{T}_i = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ I \end{bmatrix} \qquad \widetilde{\boldsymbol{u}}(k;N) = \begin{bmatrix} \overline{\boldsymbol{u}}_1(k;N) \\ \overline{\boldsymbol{u}}_2(k;N) \\ \vdots \\ \overline{\boldsymbol{u}}_M(k;N) \end{bmatrix}$$

$$\mathcal{T} = \operatorname{diag}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_M) \qquad \boldsymbol{g}(x(k)) = \begin{bmatrix} \boldsymbol{g}_1(x(k)) \\ \boldsymbol{g}_2(x(k)) \\ \vdots \\ \boldsymbol{g}_M(x(k)) \end{bmatrix}$$

with E_{ij} defined in Appendix A, and $\boldsymbol{g}_i(x(k))$ defined in (8). The definition of $\mathcal{T}_i$ is such that $x_i(k+N|k) = \mathcal{T}_i{}'\overline{\boldsymbol{x}}_i(k)$.

Let $\mathbb{X}_N$ be the set of states for which the QP (38) is feasible. We have $\mathcal{X}_N \subseteq \mathbb{X}_N \subseteq \mathcal{X}$. Constrained stabilizability, therefore, follows. The QP (38) is a centralized calculation; distributed versions for this initialization QP can be derived using techniques similar to those presented here, but are not pursued in this paper.

7.3 Algorithm

In the terminal control FC-MPC algorithm (presented below as Algorithm 3), a dummy terminal system state vector $\zeta_i^p(k;N)$ is required for each subsystem i at iterate p. The vector $\zeta_i^p(k;N), i = 1,2,\dots,M$ is constructed using the subsystem-based procedure described below.

Algorithm 2 **(Computing $\zeta_i^p(k;N)$).**
Given at iterate p, $\overline{\boldsymbol{u}}_i^{*(p)}(k;N)$ and $\overline{\boldsymbol{u}}_j^{p-1}(k;N), \forall\, j = 1,2,\dots,M, j \neq i$.
do $\forall\, l = 1,2,\dots,M$
  if $l = i$
    Calculate $\overline{\boldsymbol{x}}_i^{*(p)}(\overline{\boldsymbol{u}}_1^{p-1}(k;N), \dots, \overline{\boldsymbol{u}}_i^{*(p)}(k;N), \dots, \overline{\boldsymbol{u}}_M^{p-1}(k;N); x(k))$
    $x_i^{*(p)}(k+N|k) \leftarrow \mathcal{T}_i{}'\, \overline{\boldsymbol{x}}_i^{*(p)}(\cdot)$
  else if $l \neq i$
    Calculate dummy state trajectory
      $\overline{\boldsymbol{z}}_l^{*(p)}(\cdot) = \overline{\boldsymbol{x}}_l^{*(p)}(\overline{\boldsymbol{u}}_1^{p-1}(k;N), \dots, \overline{\boldsymbol{u}}_i^{*(p)}(k;N), \dots, \overline{\boldsymbol{u}}_M^{p-1}(k;N); x(k))$
    $z_l^{*(p)}(k+N|k) \leftarrow \mathcal{T}_l{}'\; \overline{\boldsymbol{z}}_l^{*(p)}(\cdot)$
  end (if)
end(do)
$\zeta_i^p(k;N) \leftarrow [z_1^{*(p)}(k+N|k)', \dots, z_{i-1}^{*(p)}(k+N|k)', x_i^{*(p)}(k+N|k)', z_{i+1}^{*(p)}(k+N|k)', \dots, z_M^{*(p)}(k+N|k)']'$

The following algorithm can be employed for the terminal control version of cooperation-based distributed MPC.

Algorithm 3 **(Terminal control FC-MPC)**.
Given $(x_i(k)), \mathbb{Q}_i, \mathbb{R}_i,\ \forall\, i = 1,2,\dots,M$
$p_{\max}(k) \geq 0, \epsilon > 0$ and $p \leftarrow 1$
1. Choose a finite horizon N_0. $N \leftarrow N_0$.
2. **If** $x(0) \notin \mathcal{O}_\infty$

a. Calculate $\left[\overline{\boldsymbol{u}}_1^0(k;N)', \overline{\boldsymbol{u}}_2^0(k;N)', \ldots, \overline{\boldsymbol{u}}_M^0(k;N)'\right]' \in \arg\mathcal{L}^{\mathrm{N}}(x(k))$ (Section 7.2)

b. Construct $\boldsymbol{u}_1(k), \boldsymbol{u}_2(k), \ldots, \boldsymbol{u}_M(k)$ (see Remark 3 in Section 7.1)

else
Construct $\overline{\boldsymbol{u}}_i(k;N)$ and $\boldsymbol{u}_i(k),\ i = 1, 2, \ldots, M$ (Remark 3)
end(if)

3. $\rho_i \leftarrow \Gamma\epsilon, \Gamma \gg 1$
4. **while** $\rho_i > \epsilon$ for some $i = 1, 2, \ldots, M$ and $p \leq p_{\max}$
    do $\forall\ i = 1, 2, \ldots, M$
        $\overline{\boldsymbol{u}}_i^{*(p)}(k;N) \in \arg\mathcal{P}_i^N$, (see (36))
        Calculate $\zeta_i^p(k;N)$ using Algorithm 2
        if $\zeta_i^p(k;N) \notin \mathcal{O}_\infty$
            (i) Increase N
            (ii) Extract $\overline{\boldsymbol{u}}_j^{p-1}(k;N)$ from $\boldsymbol{u}_j^{p-1}(k),\ \forall\ j = 1, 2, \ldots, M$
            (iii) Goto step 4.
        end (if)
    end (do)
    for each $i = 1, 2, \ldots, M$
        $\overline{\boldsymbol{u}}_i^p(k;N) = w_i\overline{\boldsymbol{u}}_i^{*(p)}(k;N) + (1 - w_i)\,\overline{\boldsymbol{u}}_i^{p-1}(k;N)$
        Transmit $\overline{\boldsymbol{u}}_i^p$ to each interconnected subsystem $j = 1, 2, \ldots, M, j \neq i$.
        $\rho_i = \|\overline{\boldsymbol{u}}_i^p(k;N) - \overline{\boldsymbol{u}}_i^{p-1}(k;N)\|$
    end (for)
    Construct $\boldsymbol{u}_1^p(k), \boldsymbol{u}_2^p(k), \ldots, \boldsymbol{u}_M^p(k)$ (Remark 3)
    $p \leftarrow p + 1$
end (while)

If $\zeta_i^p(k;N) \in \mathcal{O}_\infty, \forall\ i = 1, 2, \ldots, M$ then from convexity of $\mathcal{O}_\infty$ and noting from Algorithm 3 that $x^p(k+N|k) = \sum_i w_i\zeta_i^p(k;N)$, we have $x^p(k+N|k) \in \mathcal{O}_\infty$. Define the shifted input trajectory for subsystem i at time $k+1$ as

$$\overline{\boldsymbol{u}}_i^0(k+1) = \left[u_i^{p(k)}(k+1|k)', \ldots, u_i^{p(k)}(k+N-1|k)', \left(\sum_{j=1}^{M} K_{ij}x_j^{p(k)}(k+N|k)\right)'\right]' \tag{39}$$

For the nominal case, the set of shifted input trajectories (39) is a feasible set of input trajectories at time $k+1$. In the nominal case, therefore, the initialization QP (38) has to be solved only once at $k = 0$.

Lemmas 1 and 2 established for terminal penalty FC-MPC (Section 5) are also valid for terminal control FC-MPC. At convergence of the exchanged input trajectories, the performance of terminal control FC-MPC is within a pre-specified tolerance of the infinite horizon optimal (centralized, constrained LQR [26, 28]) performance. If (A, B) is stabilizable, (A, C) and $(A, \mathcal{Q}^{\frac{1}{2}})$ are detectable, and $Q_i \geq 0, R_i > 0,\ \forall\ i = 1, 2, \ldots, M$, the terminal control FC-MPC control law is nominally asymptotically stable for all values of the iteration number $p(k) > 0$.

7.4 Examples

7.4.1 Two area power system with FACTS device

We revisit the two area power system considered in Section 5.7.5. A 28% load increase affects area 1 at time 10 sec and simultaneously, an identical load disturbance affects area 2. The controller parameters are $R_1 = \text{diag}(1,1), R_2 = 1, Q_1 = \text{diag}(10,0,0,10), Q_2 = \text{diag}(10,0,0), \Delta_{\text{samp}} = 2$ sec. The controlled variables (CVs) for the MPC in area 1 are $\Delta\omega_1$ and $\Delta\delta_{12}$. The CV for the MPC in area 2 is $\Delta\omega_2$. In this case, we evaluate the load disturbance rejection performance of terminal control FC-MPC (FC-MPC (tc)) and compare it against the performance of terminal penalty FC-MPC (FC-MPC (tp)) and centralized constrained LQR (CLQR).

The relative performance of FC-MPC(tc), FC-MPC(tp) and CLQR rejecting the described load disturbance is shown in Figure 6. For terminal control FC-MPC employing Algorithm 3, an initial control horizon length (N_0) of 20 is selected. This choice of N is sufficient to steer the dummy state vectors $\zeta_i(\cdot), \forall\ i = 1,2,\ldots,M$ to $\mathcal{O}_\infty$ throughout the period where the effect of the load disturbance persists. The terminal penalty FC-MPC employs Algorithm 1 (Section 5) .

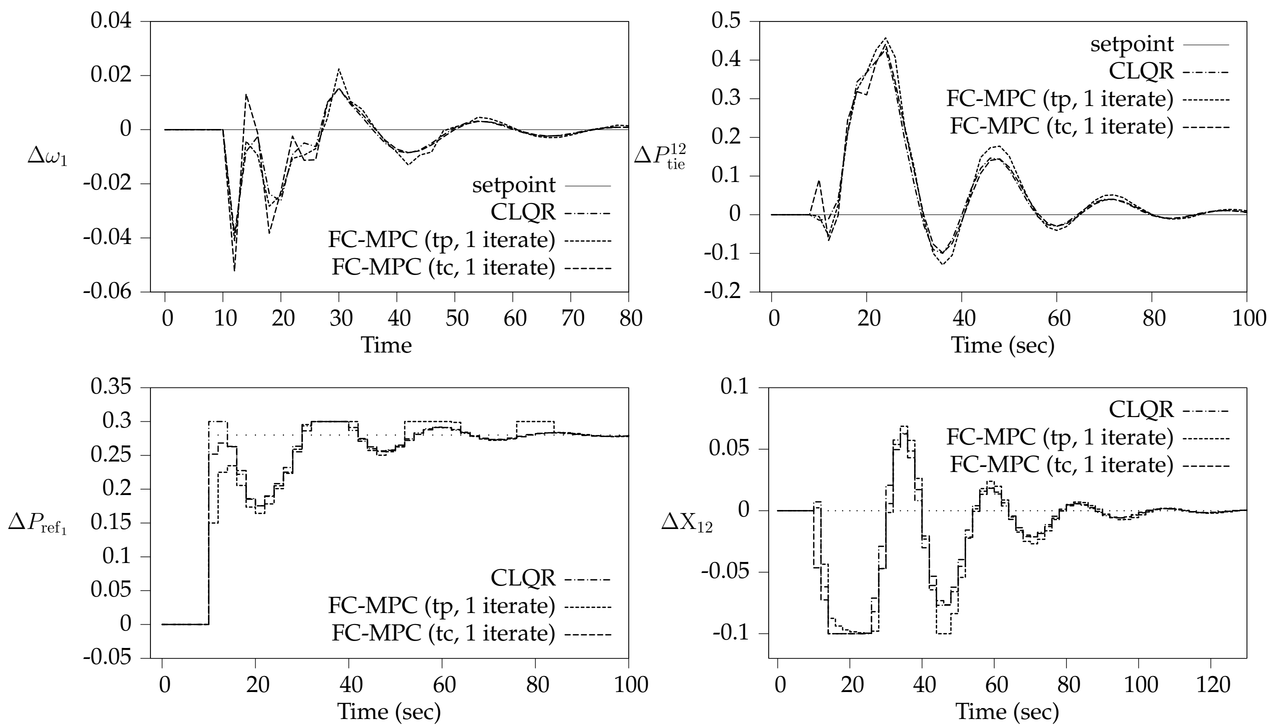

Figure 6: Comparison of load disturbance rejection performance of terminal control FC-MPC, terminal penalty FC-MPC and CLQR. Change in frequency $\Delta\omega_1$, tie-line power flow $\Delta P_{\text{tie}}^{12}$, load reference setpoints ΔP_{ref_1} and ΔP_{ref_2}.

Due to an increase in load in both control areas, the MPCs (in areas 1 and 2) order an increase in generation. In Figure 6, the transient tie-line power flow and frequency deviations under FC-MPC (tc, 1 iterate) are almost identical to the infinite horizon optimal CLQR performance. The incurred control costs are given in Table 8. FC-MPC (tc, 1 iterate) achieves a performance improvement of about $\sim 16\%$ compared to FC-MPC (tp, 1 iterate). If 5 iterates per sampling interval are permissible, the disturbance rejection performance of FC-MPC (tc, 5 iterates) is within 0.5% of CLQR performance. The performance loss incurred under FC-MPC (tp, 5 iterates), relative to CLQR performance, is about 13%, which is significantly higher than the performance loss incurred with FC-MPC (tc, 5 iterates).

Table 8: Performance of different control formulations relative to centralized constrained LQR (CLQR), $\Delta\Lambda\% = \frac{\Lambda_{\text{config}} - \Lambda_{\text{cent}}}{\Lambda_{\text{cent}}} \times 100$.

	$\Lambda \times 10^{-3}$	$\Delta\Lambda\%$
CLQR	1.77	
FC-MPC (tp, 1 iterate)	2.21	25
FC-MPC (tc, 1 iterate)	1.93	9.2
FC-MPC (tp, 5 iterates)	2	12.9
FC-MPC (tc, 5 iterates)	1.774	< 0.2

7.4.2 Unstable four area power network

Consider the four area power network described in Section 5.7.4. In this case though, $M_4^a = 40$ to force the system to be open-loop unstable. The regulator parameters are specified in Table 9. The sampling interval $\Delta_{\text{samp}} = 2$ sec. At time 10 sec, the load in area 2 increases by 15% and simultaneously, the load in area 3 decreases by 15%. The load disturbance rejection performance of terminal control FC-MPC (FC-MPC(tc)) is investigated and compared to the performance of the benchmark CLQR.

Figure 7 depicts the disturbance rejection performance of FC-MPC (tc) and CLQR. Only quantities relating to area 2 are shown as variables in other areas displayed similar qualitative behavior. The associated control costs are given in Table 10. For terminal control FC-MPC terminated after 1 iterate, the load disturbance rejection performance is within 13% of CLQR performance. If 5 iterates per sampling interval are possible, the incurred performance loss drops to $< 1.5\%$.

Table 9: Regulator parameters for unstable four area power network.

$Q_1 = \text{diag}(50, 0, 0)$	$R_1 = 1$
$Q_2 = \text{diag}(50, 0, 0, 50)$	$R_2 = 1$
$Q_3 = \text{diag}(50, 0, 0, 50)$	$R_3 = 1$
$Q_4 = \text{diag}(50, 0, 0, 50)$	$R_4 = 1$

Table 10: Performance of terminal control FC-MPC relative to centralized constrained LQR (CLQR), $\Delta\Lambda\% = \frac{\Lambda_{\text{config}} - \Lambda_{\text{cent}}}{\Lambda_{\text{cent}}} \times 100$.

	$\Lambda \times 10^{-2}$	$\Delta\Lambda\%$
CLQR	4.91	
FC-MPC (tc, 1 iterate)	5.52	12.4
FC-MPC (tc, 5 iterates)	4.97	1.2

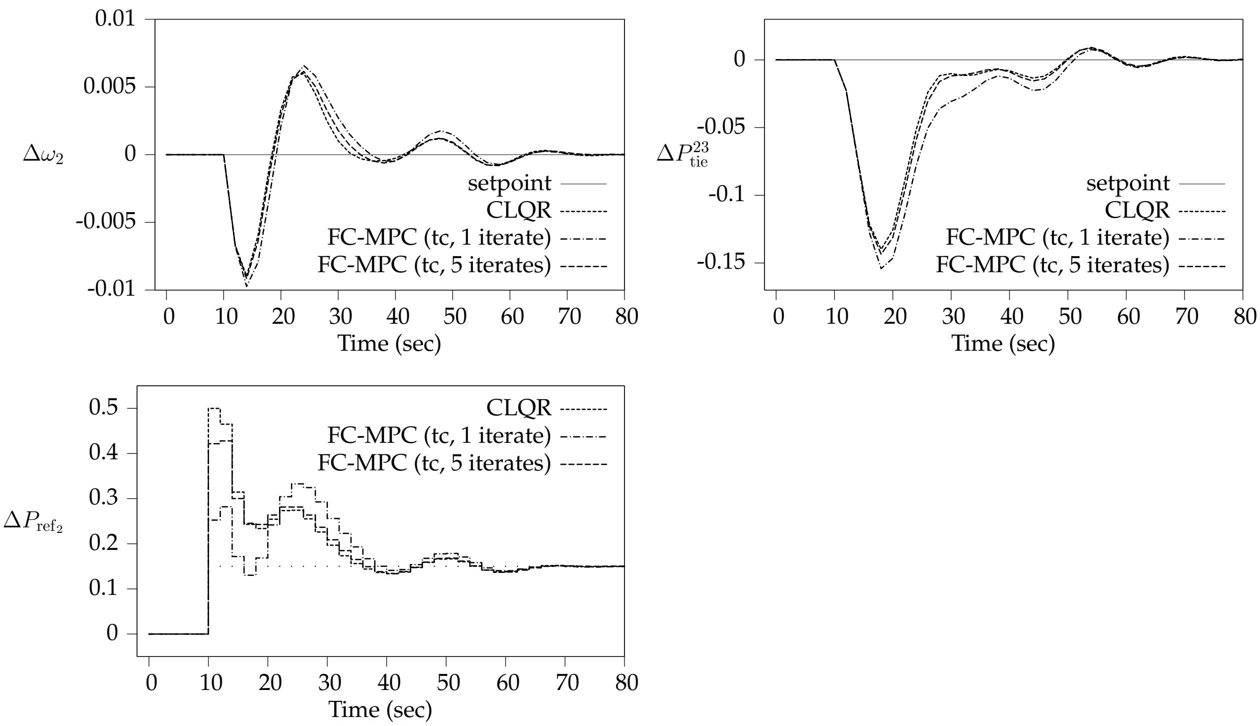

Figure 7: Performance of FC-MPC (tc) and CLQR, rejecting a load disturbance in areas 2 and 3. Change in local frequency $\Delta\omega_2$, tie-line power flow $\Delta P_{\text{tie}}^{23}$ and load reference setpoint ΔP_{ref_2}.

8 Discussion and conclusions

Centralized MPC is not well suited for control of large-scale, geographically expansive systems such as power systems. However, performance benefits obtained with centralized MPC can be realized through distributed MPC strategies. Distributed MPC strategies rely on decomposition of the overall system into interconnected subsystems, and iterative optimization and exchange of information between these subsystems. An MPC optimization problem is solved within each subsystem, using local measurements and the latest available external information (from the previous iterate).

Various forms of distributed MPC have been considered. It is shown that communication-based MPC is an unreliable strategy for systemwide control. Feasible cooperation-based MPC (FC-MPC) precludes the possibility of parochial controller behavior by forcing the MPCs to cooperate towards attaining systemwide objectives. A terminal penalty version of FC-MPC was initially established. The solution obtained at convergence of the FC-MPC algorithm is identical to the centralized MPC solution (and therefore, Pareto optimal). In addition, the FC-MPC algorithm can be terminated prior to convergence without compromising feasibility or closed-loop stability of the resulting distributed controller. This feature allows the practitioner to terminate the algorithm at the end of the sampling interval, even if convergence is not achieved.

A terminal control FC-MPC framework, which achieves infinite horizon optimal performance at convergence, has also been described. For small values of N, the performance of terminal control FC-MPC is superior to that of terminal penalty FC-MPC.

The computational overhead in terminal control FC-MPC, compared to terminal penalty FC-MPC, is in the determination of an N that steers the terminal system state inside the maximal output admissible set $\mathcal{O}_\infty$. A judicious choice of N_0 and an effective heuristic for increasing N will improve implementation efficiency. Also, $\mathcal{O}_\infty$ must be recalculated every time a setpoint change is planned. Selection of an appropriate N and determination of $\mathcal{O}_\infty$ are not, however, issues confined to distributed MPC. They are concerns in the centralized MPC framework as well [26].

An alternate strategy for terminal control FC-MPC is to explicitly enforce a terminal constraint that forces each subsystem-based estimate of the state vector (ζ_i) to be in $\mathcal{O}_\infty$. For small N, this strategy typically leads to excessively aggressive controller response, which is undesirable. Enforcing the terminal set constraint $\zeta_i(\cdot) \in \mathcal{O}_\infty,\ i = 1, 2, \ldots, M$ explicitly results in a coupled input constraint. For this formulation, feasibility and stability of the resulting control law can be proved. Optimality at convergence, however, is not necessarily obtained.

Examples were presented to illustrate the applicability and effectiveness of the proposed distributed MPC framework for automatic generation control (AGC). First, a two area network was considered. Both communication-based MPC and cooperation-based MPC outperformed AGC due to their ability to handle process constraints. The controller defined by terminating Algorithm 1 after 5 iterates achieves performance that is almost identical to centralized MPC. Next, the performance of the different MPC frameworks are evaluated for a four area network. For this case, communication-based MPC leads to closed-loop instability. FC-MPC (1 iterate) stabilizes the system and achieves performance that is within 26% of centralized MPC performance. The two area network considered earlier, with an additional FACTS device to control tie line impedence, is examined subsequently. Communication-based MPC stabilizes the system but gives unacceptable closed-loop performance. The FC-MPC framework is shown to allow coordination of FACTS controls with AGC. The controller defined by terminating Algorithm 1 after just 1 iterate gives an $\sim 190\%$ improvement in performance compared to communication-based MPC. For this case, therefore, the cooperative aspect of FC-MPC was very important for achieving acceptable response. Next, the two area network with FACTS device was used to compare the performance of terminal penalty FC-MPC and terminal control FC-MPC. As expected, terminal control FC-MPC outperforms terminal penalty FC-MPC for short horizon lengths. Finally, the performance of terminal control FC-MPC is evaluated on an unstable four area network. FC-MPC (tc, 5 iterates) achieves performance that is within 1.5% of the centralized constrained LQR performance.

9 ACKNOWLEDGMENT

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A Model Manipulation

To ensure strict feasibility of the FC-MPC algorithm, it is convenient to eliminate the states $\overline{\boldsymbol{x}}_i$, $i = 1, 2, \ldots, M$ using the PM (1). Propagating the model for each subsystem through the control horizon N gives

$$\overline{\boldsymbol{x}}_i = \overline{E}_{ii}\overline{\boldsymbol{u}}_i + \overline{f}_{ii}x_i(k) + \sum_{j\neq i}[\overline{E}_{ij}\overline{\boldsymbol{u}}_j + \overline{g}_{ij}\overline{\boldsymbol{x}}_j + \overline{f}_{ij}x_j(k)] \qquad \forall\ i = 1, 2, \ldots, M \tag{40}$$

in which

$$\overline{E}_{ij} = \begin{bmatrix} B_{ij} & 0 & \dots & \dots & 0 \\ A_{ii}B_{ij} & B_{ij} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{ii}^{N-1}B_{ij} & \dots & \dots & \dots & B_{ij} \end{bmatrix} \qquad \overline{f}_{ij} = \begin{bmatrix} A_{ij} \\ A_{ii}A_{ij} \\ \vdots \\ A_{ii}^{N-1}A_{ij} \end{bmatrix}$$

$$\overline{g}_{ij} = \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ A_{ij} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{ii}^{N-2}A_{ij} & A_{ii}^{N-3}A_{ij} & \dots & \dots & 0 \end{bmatrix}.$$

Combining the models in (40), $\forall\, i = 1, 2, \dots, M$, gives the following system of equations

$$\mathcal{A}\widetilde{\boldsymbol{x}} = \mathcal{E}\widetilde{\boldsymbol{u}} + \mathcal{G}x(k) \tag{41}$$

in which

$$\mathcal{G} = \begin{bmatrix} \overline{f}_{11} & \overline{f}_{12} & \dots & \overline{f}_{1M} \\ \overline{f}_{21} & \overline{f}_{22} & \dots & \overline{f}_{2M} \\ \ddots & \ddots & \ddots & \ddots \\ \overline{f}_{M1} & \dots & \dots & \overline{f}_{MM} \end{bmatrix} \qquad \mathcal{E} = \begin{bmatrix} \overline{E}_{11} & \overline{E}_{12} & \dots & \overline{E}_{1M} \\ \overline{E}_{21} & \overline{E}_{22} & \dots & \overline{E}_{2M} \\ \ddots & \ddots & \ddots & \ddots \\ \overline{E}_{M1} & \dots & \dots & \overline{E}_{MM} \end{bmatrix}$$

$$\mathcal{A} = \begin{bmatrix} I & -\overline{g}_{12} & \dots & -\overline{g}_{1M} \\ -\overline{g}_{21} & I & \dots & -\overline{g}_{2M} \\ \ddots & \ddots & \ddots & \ddots \\ -\overline{g}_{M1} & \dots & \dots & I \end{bmatrix} \quad \widetilde{\boldsymbol{x}} = \begin{bmatrix} \overline{\boldsymbol{x}}_1 \\ \overline{\boldsymbol{x}}_2 \\ \vdots \\ \overline{\boldsymbol{x}}_M \end{bmatrix} \quad \widetilde{\boldsymbol{u}} = \begin{bmatrix} \overline{\boldsymbol{u}}_1 \\ \overline{\boldsymbol{u}}_2 \\ \vdots \\ \overline{\boldsymbol{u}}_M \end{bmatrix} \tag{42}$$

Since the system is LTI, a solution to the system (41) exists for each permissible RHS. Matrix $\mathcal{A}$ is therefore invertible and consequently, we can write for each $i = 1, 2, \dots, M$

$$\overline{\boldsymbol{x}}_i = E_{ii}\overline{\boldsymbol{u}}_i + f_{ii}x_i(k) + \sum_{j \neq i}[E_{ij}\overline{\boldsymbol{u}}_j + f_{ij}x_j(k)]. \tag{43}$$

in which E_{ij} and f_{ij}, $\forall\, j = 1, 2, \dots, M$ denote the appropriate partitions of $\mathcal{A}^{-1}\mathcal{E}$ and $\mathcal{A}^{-1}\mathcal{G}$ respectively.

B Terminal penalty FC-MPC

Lemma 3 (Minimum principle for constrained, convex optimization)**.** *Let $\mathcal{X}$ be a convex set and let f be a convex function over $\mathcal{X}$. A necessary and sufficient condition for x^* to be a global minimum of f over $\mathcal{X}$ is*

$$\nabla f(x^*)'(x - x^*) \geq 0, \; \forall\, x \in \mathcal{X}$$

A proof is given in [4, p. 194]

Proof of Lemma 1. From Algorithm 1 we know that

$$\Phi\left(\boldsymbol{u}_1^{p-1},\dots \boldsymbol{u}_{i-1}^{p-1},\boldsymbol{u}_i^{*(p)},\boldsymbol{u}_{i+1}^{p-1},\dots,\boldsymbol{u}_M^{p-1};x(k)\right) \leq \Phi\left(\boldsymbol{u}_1^{p-1},\boldsymbol{u}_2^{p-1},\dots,\boldsymbol{u}_M^{p-1};x(k)\right) \quad \forall\, i=1,2,\dots,M \tag{44}$$

Therefore, from the definition of $\boldsymbol{u}_i^p$ (Algorithm 1) we have

$$\begin{aligned}\Phi\left(\boldsymbol{u}_1^{p},\boldsymbol{u}_2^{p},\dots,\boldsymbol{u}_M^{p};x(k)\right) &= \Phi\left(w_1\boldsymbol{u}_1^{*(p)}+(1-w_1)\boldsymbol{u}_1^{p-1},\dots,w_M\boldsymbol{u}_M^{*(p)}+(1-w_M)\boldsymbol{u}_M^{p-1};x(k)\right)\\ &= \Phi\Big(w_1\boldsymbol{u}_1^{*(p)}+w_2\boldsymbol{u}_1^{p-1}+\ldots+w_M\boldsymbol{u}_1^{p-1},w_1\boldsymbol{u}_2^{p-1}+w_2\boldsymbol{u}_2^{*(p)}+\ldots+w_M\boldsymbol{u}_2^{p-1},\\ &\qquad \ldots\ldots,w_1\boldsymbol{u}_M^{p-1}+w_2\boldsymbol{u}_M^{p-1}+\ldots+\boldsymbol{u}_M^{*(p)};x(k)\Big)\end{aligned}$$

By convexity of $\Phi(\cdot)$,

$$\begin{aligned}&\leq \sum_{r=1}^{M} w_r\Phi\left(\boldsymbol{u}_1^{p-1},\dots,\boldsymbol{u}_{r-1}^{p-1},\boldsymbol{u}_r^{*(p)},\boldsymbol{u}_{r+1}^{p-1},\dots,\boldsymbol{u}_M^{p-1};x(k)\right)\\ &\leq \sum_{r=1}^{M} w_r\Phi\left(\boldsymbol{u}_1^{p-1},\dots,\boldsymbol{u}_{r-1}^{p-1},\boldsymbol{u}_r^{p-1},\boldsymbol{u}_{r+1}^{p-1},\dots,\boldsymbol{u}_M^{p-1};x(k)\right)\\ &= \Phi\left(\boldsymbol{u}_1^{p-1},\boldsymbol{u}_2^{p-1},\dots,\boldsymbol{u}_M^{p-1};x(k)\right)\end{aligned} \tag{45}$$

in which equality is obtained if $\boldsymbol{u}_i^p = \boldsymbol{u}_i^{p-1}, \forall\, i=1,2,\dots,M$. □

Proof of Lemma 2. Since the level set

$$S_0 = \{(\overline{\boldsymbol{u}}_1,\overline{\boldsymbol{u}}_2,\dots,\overline{\boldsymbol{u}}_M) \mid \Phi(\boldsymbol{u}_1,\boldsymbol{u}_2,\dots,\boldsymbol{u}_M;x(k)) \leq \Phi(\boldsymbol{u}_1^0,\boldsymbol{u}_2^0,\dots,\boldsymbol{u}_M^0;x(k))\}$$

is closed and bounded (hence compact), a limit point for Algorithm 1 exists. We have that $(\boldsymbol{u}_1^*,\dots,\boldsymbol{u}_M^*)$ is the unique solution for the centralized MPC optimization problem (Equation (11)). Let $\Phi^* = \Phi(\boldsymbol{u}_1^*,\boldsymbol{u}_2^*,\dots,\boldsymbol{u}_M^*;x(k))$. Define $\boldsymbol{u}_i^\infty = [\overline{\boldsymbol{u}}_i^{\infty\prime},0,0,\dots]'$. Assume that the sequence $(\overline{\boldsymbol{u}}_1^p,\overline{\boldsymbol{u}}_2^p,\dots,\overline{\boldsymbol{u}}_M^p)$, generated by Algorithm 1, converges to a feasible subset of the nonoptimal level set

$$S_\infty = \{(\overline{\boldsymbol{u}}_1,\overline{\boldsymbol{u}}_2,\dots,\overline{\boldsymbol{u}}_M) \mid \Phi(\boldsymbol{u}_1,\boldsymbol{u}_2,\dots,\boldsymbol{u}_M;x(k)) = \Phi^\infty\}$$

Since $\Phi(\cdot)$ is strictly convex and by assumption of nonoptimality $\Phi^\infty > \Phi^*$. Let $(\overline{\boldsymbol{u}}_1^\infty,\dots,\overline{\boldsymbol{u}}_M^\infty) \in S_\infty$ be generated by Algorithm 1 for p large. To establish convergence of Algorithm 1 to a point rather than a limit set, we assume the contrary and show a contradiction. Suppose that Algorithm 1 does not converge to a point. Our assumption here implies that there exists $(\overline{\boldsymbol{v}}_1,\dots,\overline{\boldsymbol{v}}_M) \in S_\infty$ generated by the next iterate of Algorithm 1 with $(\overline{\boldsymbol{v}}_1,\dots,\overline{\boldsymbol{v}}_M) \neq (\overline{\boldsymbol{u}}_1^\infty,\dots,\overline{\boldsymbol{u}}_M^\infty)$.

Consider the set of optimization problems

$$\boldsymbol{z}_i^\infty = \arg\min_{\boldsymbol{u}_i} \ \Phi(\boldsymbol{u}_1^\infty,\dots,\boldsymbol{u}_{i-1}^\infty,\boldsymbol{u}_i,\boldsymbol{u}_{i+1}^\infty,\dots,\boldsymbol{u}_M^\infty;x(k)) \tag{46a}$$

$$u_i(l|k) \in \Omega_i,\ 0 \leq l \leq N-1 \tag{46b}$$

$$u_i(l|k) = 0,\ N \leq l \tag{46c}$$

$$\forall\, i=1,2,\dots,M$$

We have $\boldsymbol{z}_i^\infty = [\overline{\boldsymbol{z}}_i^{\infty\prime}, 0, 0, \ldots]'$ in which $\overline{\boldsymbol{z}}_i^\infty = [z_i^\infty(0)', \ldots, z_i^\infty(N-1)']'$. By assumption, there exists at least one i for which $\boldsymbol{z}_i^\infty \neq \boldsymbol{u}_i^\infty$. WLOG let $\boldsymbol{z}_1^\infty \neq \boldsymbol{u}_1^\infty$. By definition, $\overline{\boldsymbol{v}}_i = w_i \overline{\boldsymbol{z}}_i^\infty + (1-w_i)\overline{\boldsymbol{u}}_i^\infty, \ \forall\, i = 1, 2, \ldots, M$. It follows that $\boldsymbol{v}_i = [\overline{\boldsymbol{v}}_i', 0, 0, \ldots]'$. Since $(\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2, \ldots, \overline{\boldsymbol{v}}_M) \in S_\infty$, $\Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M; x(k)) = \Phi^\infty$. Using convexity of $\Phi(\cdot)$, we have

$$\begin{aligned}
\Phi^\infty = \Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M; x(k)) &= \Phi(w_1 \boldsymbol{z}_1^\infty + (1-w_1)\boldsymbol{u}_1^\infty, \ldots, w_M \boldsymbol{z}_M^\infty + (1-w_M)\boldsymbol{u}_M^\infty; x(k)) \\
&< w_1 \Phi(\boldsymbol{z}_1^\infty, \boldsymbol{u}_2^\infty, \ldots, \boldsymbol{u}_M^\infty; x(k)) + \ldots \\
&\qquad \ldots + w_M \Phi(\boldsymbol{u}_1^\infty, \ldots, \boldsymbol{u}_{M-1}^\infty, \boldsymbol{z}_M^\infty; x(k)) \\
&< w_1 \Phi^\infty + \ldots w_M \Phi^\infty \\
&= \Phi^\infty
\end{aligned}$$

in which the strict inequality follows from $\boldsymbol{z}_i^\infty \neq \boldsymbol{u}_i^\infty$ for at least one $i = 1, 2, \ldots, M$. Hence, a contradiction.

Suppose now that $(\overline{\boldsymbol{u}}_1^p, \overline{\boldsymbol{u}}_2^p, \ldots, \overline{\boldsymbol{u}}_M^p) \to (\overline{\boldsymbol{u}}_1^\infty, \overline{\boldsymbol{u}}_2^\infty, \ldots, \overline{\boldsymbol{u}}_M^\infty) \neq (\overline{\boldsymbol{u}}_1^*, \overline{\boldsymbol{u}}_2^*, \ldots, \overline{\boldsymbol{u}}_M^*)$. Since the optimizer is unique, $\Phi(\boldsymbol{u}_1^*, \boldsymbol{u}_2^*, \ldots, \boldsymbol{u}_M^*; x(k)) < \Phi(\boldsymbol{u}_1^\infty, \boldsymbol{u}_2^\infty, \ldots, \boldsymbol{u}_M^\infty; x(k))$.

Since $(\overline{\boldsymbol{u}}_1^p, \overline{\boldsymbol{u}}_2^p, \ldots, \overline{\boldsymbol{u}}_M^p)$, generated using Algorithm 1, converges to $(\overline{\boldsymbol{u}}_1^\infty, \overline{\boldsymbol{u}}_2^\infty, \ldots, \overline{\boldsymbol{u}}_M^\infty)$, we have

$$\boldsymbol{u}_i^\infty = \arg\min_{\boldsymbol{u}_i} \ \Phi(\boldsymbol{u}_1^\infty, \ldots, \boldsymbol{u}_{i-1}^\infty, \boldsymbol{u}_i, \boldsymbol{u}_{i+1}^\infty, \ldots, \boldsymbol{u}_M^\infty; x(k)) \tag{47a}$$

$$u_i(l|k) \in \Omega_i, \ 0 \leq l \leq N-1 \tag{47b}$$

$$u_i(l|k) = 0, \ N \leq l \tag{47c}$$

$$\forall\, i = 1, 2, \ldots, M$$

From Lemma 3,

$$\nabla_{\overline{\boldsymbol{u}}_j} \Phi(\boldsymbol{u}_1^\infty, \ldots, \boldsymbol{u}_M^\infty; x(k))'(\overline{\boldsymbol{u}}_j^* - \overline{\boldsymbol{u}}_j^\infty) \geq 0 \qquad \forall\, j = 1, 2, \ldots, M$$

Define $\Delta\overline{\boldsymbol{u}}_j = \overline{\boldsymbol{u}}_j^* - \overline{\boldsymbol{u}}_j^\infty$ and $\Delta\boldsymbol{u}_j = \boldsymbol{u}_j^* - \boldsymbol{u}_j^\infty = [\Delta\overline{\boldsymbol{u}}_j', 0, 0, \ldots]' \quad \forall\, j = 1, 2, \ldots, M$. We have, from our assumption $(\overline{\boldsymbol{u}}_1^\infty, \overline{\boldsymbol{u}}_2^\infty, \ldots, \overline{\boldsymbol{u}}_M^\infty) \neq (\overline{\boldsymbol{u}}_1^*, \overline{\boldsymbol{u}}_2^*, \ldots, \overline{\boldsymbol{u}}_M^*)$, that $\Delta\overline{\boldsymbol{u}}_i \neq 0$ for at least one index i, $1 \leq i \leq M$.

A second order Taylor series expansion around $(\boldsymbol{u}_1^\infty, \boldsymbol{u}_2^\infty, \ldots, \boldsymbol{u}_M^\infty)$ gives

$$\begin{aligned}
\Phi(\boldsymbol{u}_1^*, \boldsymbol{u}_2^*, \ldots, \boldsymbol{u}_M^*; x(k)) &= \Phi(\boldsymbol{u}_1^\infty + \Delta\boldsymbol{u}_1, \boldsymbol{u}_2^\infty + \Delta\boldsymbol{u}_2, \ldots, \boldsymbol{u}_M^\infty + \Delta\boldsymbol{u}_M; x(k)) \\
&= \Phi(\boldsymbol{u}_1^\infty, \ldots, \boldsymbol{u}_M^\infty; x(k)) + \underbrace{\sum_{j=1}^{M} \nabla_{\overline{\boldsymbol{u}}_j} \Phi(\boldsymbol{u}_1^\infty, \ldots, \boldsymbol{u}_M^\infty; x(k))' \Delta\overline{\boldsymbol{u}}_j}_{\geq 0, \text{ Lemma 3}} \\
&\quad + \underbrace{\frac{1}{2} \begin{bmatrix} \Delta\overline{\boldsymbol{u}}_1 \\ \vdots \\ \Delta\overline{\boldsymbol{u}}_M \end{bmatrix}' \nabla^2 \Phi(\boldsymbol{u}_1^\infty, \ldots, \boldsymbol{u}_M^\infty; x(k)) \begin{bmatrix} \Delta\overline{\boldsymbol{u}}_1 \\ \vdots \\ \Delta\overline{\boldsymbol{u}}_M \end{bmatrix}}_{\geq 0, \text{ since } \Phi(\cdot) \text{ p.d. quadratic}}
\end{aligned} \tag{48}$$

Using (48) and optimality of $(\boldsymbol{u}_1^*, \boldsymbol{u}_2^*, \ldots, \boldsymbol{u}_M^*)$ gives

$$\Phi(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_M^*; x(k)) = \Phi(\boldsymbol{u}_1^\infty, \ldots, \boldsymbol{u}_M^\infty; x(k)) + \beta(\Delta \boldsymbol{u}_1, \ldots, \Delta \boldsymbol{u}_M) \leq \Phi(\boldsymbol{u}_1^\infty, \ldots, \boldsymbol{u}_M^\infty; x(k)) \quad (49)$$

in which $\beta(\cdot)$ is a positive definite function (from (48)). We have from (49) that $\beta(\cdot) \leq 0$, which implies $\beta(\Delta \boldsymbol{u}_1, \ldots, \Delta \boldsymbol{u}_M) = 0$. It follows, therefore, that $\overline{\boldsymbol{u}}_j^\infty = \overline{\boldsymbol{u}}_j^*, \ \forall\, j = 1, 2, \ldots, M$. Using the above relation gives $\boldsymbol{u}_j^\infty = [\overline{\boldsymbol{u}}_j^{\infty\prime}, 0, 0, \ldots]' = \boldsymbol{u}_j^*, \ \forall\, j = 1, 2, \ldots, M$.

Hence, $\Phi(\boldsymbol{u}_1^*, \boldsymbol{u}_2^*, \ldots, \boldsymbol{u}_M^*; x(k)) = \Phi(\boldsymbol{u}_1^\infty, \boldsymbol{u}_2^\infty, \ldots, \boldsymbol{u}_M^\infty; x(k))$. □

Lemma 4. *Let the input constraints in (8) be specified in terms of a collection of linear inequalities. Consider the closed ball $B_\varepsilon(0)$, in which $\varepsilon > 0$ is chosen such that the input constraints in each FC-MPC optimization problem (8) are inactive for each $x(k) \in B_\varepsilon(0)$. The distributed MPC control law defined by the FC-MPC formulation of Theorem 1 is a Lipschitz continuous function of $x(k)$ for all $x(k) \in B_\varepsilon(0)$.*

Proof. Since $0 \in \text{int}(\Omega_1 \times \Omega_2 \times \ldots \times \Omega_M)$ and the origin is Lyapunov stable and attractive with the cost relationship given by (16), it is possible to choose $\varepsilon > 0$ such that for any $x(k) \in B_\varepsilon(0)$, input constraints in the optimization problem $\mathcal{F}_i, \ \forall\, i = 1, 2, \ldots, M$ (8) are inactive.

Consider the FC-MPC optimization problem of (8). Since $\mathfrak{R}_i > 0$, the solution to the FC-MPC optimization problem is unique. The parameters that vary in the data are $x(k)$ and the input trajectories $[\overline{\boldsymbol{u}}_1^{p(k)-1}, \overline{\boldsymbol{u}}_2^{p(k)-1}, \ldots, \overline{\boldsymbol{u}}_{i-1}^{p(k)-1}, \overline{\boldsymbol{u}}_{i+1}^{p(k)-1}, \ldots, \overline{\boldsymbol{u}}_M^{p(k)-1}]$.

For subsystem i, let $\overline{\boldsymbol{u}}_i^{*(p)}(\cdot)$ represent the solution to (8) at iterate p. Let $x(k), z(k) \in B_\varepsilon(0)$. By construction of $B_\varepsilon(0)$, none of the input constraints are active. By definition $\boldsymbol{u}_i^{*(p)} = [\overline{\boldsymbol{u}}_i^{*(p)\prime}, 0, 0 \ldots]'$. Invoking [14, Theorem 3.1], $\exists\, \rho < \infty$ such that

$$\|\overline{\boldsymbol{u}}_i^{*(p)}(\cdot\,; x(k)) - \overline{\boldsymbol{u}}_i^{*(p)}(\cdot\,; z(k))\| \leq \rho \left[\|x(k) - z(k)\|^2 + \sum_{j \neq i}^{M} \|\overline{\boldsymbol{u}}_j^{p-1}(x(k)) - \overline{\boldsymbol{u}}_j^{p-1}(z(k))\|^2 \right]^{1/2} \quad (50)$$

Using the definitions of $\boldsymbol{u}_i^p(\cdot), \overline{\boldsymbol{u}}_i^p(\cdot)$, and from Algorithm 1, we have

$$\begin{aligned}
\|\boldsymbol{u}_i^{p(k)}(x(k)) - \boldsymbol{u}_i^{p(k)}(z(k))\| &= \|\overline{\boldsymbol{u}}_i^{p(k)}(x(k)) - \overline{\boldsymbol{u}}_i^{p(k)}(z(k))\| \\
&\leq w_i \|\overline{\boldsymbol{u}}_i^{*(p)}(\cdot\,; x(k)) - \overline{\boldsymbol{u}}_i^{*(p)}(\cdot\,; z(k))\| \\
&\quad + (1 - w_i)\, \|\overline{\boldsymbol{u}}_i^{p(k)-1}(x(k)) - \overline{\boldsymbol{u}}_i^{p(k)-1}(z(k))\| \\
&\leq \rho w_i \left[\|x(k) - z(k)\|^2 + \sum_{j \neq i}^{M} \|\overline{\boldsymbol{u}}_j^{p(k)-1}(x(k)) - \overline{\boldsymbol{u}}_j^{p(k)-1}(z(k))\|^2 \right]^{1/2} \\
&\quad + (1 - w_i)\, \|\overline{\boldsymbol{u}}_i^{p(k)-1}(x(k)) - \overline{\boldsymbol{u}}_i^{p(k)-1}(z(k))\|, \ \forall\, k \geq 0, \ p(k) \geq 1
\end{aligned} \quad (51)$$

At time $k = 0$, $\boldsymbol{u}_i^0 = [0, 0, \ldots]', \ \forall\, i = 1, 2, \ldots, M$, independent of the system state. It follows by induction that $\overline{\boldsymbol{u}}_i^{p(0)}(x(0))$ is a Lipschitz continuous function of $x(0)$ for all $p(0) > 0$. For $k > 0$, we have

$$\overline{\boldsymbol{u}}_i^0(x(k)) = \left[u_i^{p(k-1)}(k; x(k-1)), u_i^{p(k-1)}(k+1; x(k-1)), \ldots, u_i^{p(k-1)}(k+N-2; x(k-1)), 0 \right]$$

Since the models are causal, $\overline{\boldsymbol{u}}_i^0$ is independent of $x(k)$. Subsequently, using (51) and by induction, $\overline{\boldsymbol{u}}_i^{p(k)}(x(k))$ is Lipschitz continuous w.r.t $x(k)$ for any $p(k) > 0$ and all $k \geq 0$. By definition, $\boldsymbol{u}_i^{p(k)}(x(k)) = [\overline{\boldsymbol{u}}_i^{p(k)}(x(k))', 0, 0, \ldots]'$, $\forall\, k \geq 0$. Hence, $\boldsymbol{u}_i^{p(k)}(x(k))$ is a Lipschitz continuous function of $x(k)$ for all $k \geq 0$ and any $p(k) > 0$. In Algorithm 1, if $0 < p_{\max}(k) \leq p^* < \infty$, $\forall\, k \geq 0$, a global Lipschitz constant can be estimated. □

Proof of Theorem 1. Since $\mathcal{Q} > 0$ and A is stable, $P > 0$ [27]. The constrained stabilizable set $\mathcal{X}$ for the system is $\mathbb{R}^n$. To prove exponential stability, we use the value function $J_N^{p(k)}(x(k))$ as a candidate Lyapunov function. We need to show [30, p. 267] that there exists constants $a, b, c > 0$ such that

$$a\|x(k)\|^2 \leq J_N^p(x(k)) \leq b\|x(k)\|^2 \tag{52a}$$

$$\Delta J_N^p(x(k)) \leq -c\|x(k)\|^2 \tag{52b}$$

in which $\Delta J_N^{p(k)}(x(k)) = J_N^{p(k+1)}\left(x(k+1)\right) - J_N^{p(k)}\left(x(k)\right)$.

Let $\varepsilon > 0$ be chosen such that the input constraints remain inactive for $x \in B_\varepsilon(0)$. Such an ε exists because the origin is Lyapunov stable and $0 \in \operatorname{int}(\Omega_1 \times \ldots \Omega_M)$. Since $\Omega_i, \forall\, i = 1, 2, \ldots, M$ is compact, there exists $\sigma > 0$ such that $\|\overline{\boldsymbol{u}}_i\| \leq \sigma$. For any x satisfying $\|x\| > \varepsilon$, $\|\overline{\boldsymbol{u}}_i\| < \frac{\sigma}{\varepsilon}\|x\|$, $\forall\, i = 1, 2, \ldots, M$. For $x(k) \in B_\varepsilon(0)$, we have from Lemma 4 that $\overline{\boldsymbol{u}}_i^{p(k)}(x(k))$ is a Lipschitz continuous function of $x(k)$. There exists, therefore, a constant $\rho > 0$ such that $\|\overline{\boldsymbol{u}}_i^{p(k)}(x(k))\| \leq \rho\|x(k)\|$, $\forall\, 0 < p(k) \leq p^*$. Define $K_u = \max\left(\frac{\sigma}{\varepsilon}, \rho\right)^2$, in which $K_u > 0$ and independent of $x(k)$. The above definition gives $\|u_i^{p(k)}(k+j; x(k))\| \leq \sqrt{K_u}\|x(k)\|$, $\forall\, i = 1, 2, \ldots, M, k \geq 0$ and all $0 < p(k) \leq p_{\max}(k)$. For $j \geq 0$, define $u(k+j|k) = [u_1^{p(k)}(k+j; x(k))', \ldots, u_M^{p(k)}(k+j; x(k))']'$. By definition, $u(k|k) \equiv u(k)$. We have $\|u(k+j|k)\| = \sqrt{\sum_{i=1}^M \|u_i^{p(k)}(k+j; x(k))\|^2} \leq \sqrt{K_u M}\|x(k)\|$. Similarly, define $x(k+j|k) = [x_1^{p(k)}(k+j|k)', \ldots, x_M^{p(k)}(k+j|k)']'$, $\forall\, j \geq 0$. By definition $x(k|k) \equiv x(k)$.

Since A is stable, there exists $\overline{c} > 0$ such that $\|A^j\| \leq \overline{c}\lambda^j$ [17, Corollary 5.6.13, p. 199], in which $\lambda_{\max}(A) \leq \lambda < 1$. Hence,

$$\begin{aligned}
\|x(k+j|k)\| &\leq \|A^j\|\|x(k)\| + \sum_{l=0}^{j-1} \|A^{j-1-l}\|\|B\|\|u(k+l|k)\| \\
&\leq \overline{c}\lambda^j\|x(k)\| + \sum_{l=0}^{j-1} \overline{c}\lambda^{j-1-l}\|B\|\|u(k+l|k)\| \\
&\leq \overline{c}\left(1 + \frac{\|B\|}{1-\lambda}\sqrt{MK_u}\right)\|x(k)\|,\ \forall\, j > 0,
\end{aligned}$$

since $\sum_{l=0}^{j} \lambda^l \leq \sum_{l=0}^{\infty} \lambda^l = \frac{1}{1-\lambda}$, $\forall\, j \geq 0$. Let $\mathcal{R} = \operatorname{diag}(w_1R_1, w_2R_2, \ldots, w_MR_M)$ and $\Gamma =$

$\left[\overline{c}\left(1+\frac{\|B\|}{1-\lambda}\sqrt{MK_u}\right)\right]^2$.

$$\begin{aligned}
J_N^{p(k)}(x(k)) &= \frac{1}{2}\sum_{i=1}^{M} w_i \sum_{j=0}^{\infty}\left[\|x_i^{p(k)}(k+j|k)\|_{Q_i}^2 + \|u_i^{p(k)}(k+j|k)\|_{R_i}^2\right] \\
&= \frac{1}{2}\sum_{j=0}^{N-1}\left[x(k+j|k)'\mathcal{Q}x(k+j|k) + u(k+j|k)'\mathcal{R}u(k+j|k)\right] \\
&\qquad + \frac{1}{2}x(k+N|k)'Px(k+N|k) \\
&\le \frac{1}{2}\Bigg[\sum_{j=0}^{N-1}\left(\lambda_{\max}(\mathcal{Q})\|x(k+j|k)\|^2 + \lambda_{\max}(\mathcal{R})\|u(k+j|k)\|^2\right) \\
&\qquad + \lambda_{\max}(P)\|x(k+N|k)\|^2\Bigg] \\
&\le \frac{1}{2}\left[N\lambda_{\max}(\mathcal{Q})\Gamma + N\lambda_{\max}(\mathcal{R})K_uM + \lambda_{\max}(P)\Gamma\right]\|x(k)\|^2 \\
&\le b\|x(k)\|^2
\end{aligned}$$

in which $0 < \frac{1}{2}\left[N\lambda_{\max}(\mathcal{Q})\Gamma + N\lambda_{\max}(\mathcal{R})K_uM + \lambda_{\max}(P)\Gamma\right] \le b$.

Also, $\frac{1}{2}\lambda_{\min}(\mathcal{Q})\|x(k)\|^2 \le J_N^{p(k)}(x(k))$ Furthermore,

$$\begin{aligned}
J_N^{p(k+1)}(x(k+1)) - J_N^{p(k)}(x(k)) &\le J_N^0(x(k+1)) - J_N^{p(k)}(x(k)) \\
&= -\sum_{i=1}^{M} w_iL_i\left(x_i(k), u_i^{p(k)}(k;x(k))\right) \\
&\le -\sum_{i=1}^{M} w_iL_i\left(x_i(k), 0\right) \\
&= -\frac{1}{2}x(k)'\mathcal{Q}x(k) \\
&\le -\frac{1}{2}\lambda_{\min}(\mathcal{Q})\|x(k)\|^2
\end{aligned} \tag{53}$$

which proves the theorem. □