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Stability and optimality of distributed, linear model predictive control* Part II: Output Feedback

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Abstract

The distributed model predictive control (MPC) framework described in [38] is extended to handle output feedback. A distributed estimator design strategy is proposed. Each estimator is stable and uses only local measurements to estimate subsystem states. The feasible cooperation-based MPC (FC-MPC) algorithm presented in [38] is used for distributed regulation. Feasibility and closed-loop stability for all (FC-MPC algorithm) iteration numbers are established for the combined distributed estimator-distributed regulator in the case of decaying estimate error. A subsystem-based disturbance modeling framework to eliminate steady-state offset due to modeling errors and unmeasured disturbances is next presented. Conditions to verify suitability of chosen local disturbance models are provided. A distributed target calculation algorithm to compute steady-state targets locally is presented. All iterates generated by the distributed target calculation algorithm are feasible steady states. Conditions under which the FC-MPC framework, with distributed estimation, distributed target calculation and distributed regulation, achieves offset-free control at steady state are described. The effectiveness of the proposed framework is illustrated through two examples - a chemical plant and an irrigation canal network.

Keywords: distributed MPC, output feedback, large-scale MPC,

1 Introduction

Large, engineering systems typically consist of a number of subsystems that interact with each other as a result of material/energy and/or information flows. Many of these subsystems employ high performance control techniques such as model predictive control (MPC) to maximize local objectives. Several recent articles have studied the impact of MPC technology in the chemical industry sector [25; 41]. In many cases, tighter local control results in more pronounced interactions among the subsystems. In the decentralized control framework, the interactions among subsystems are assumed negligible and are ignored. Subsequently, decentralized control is not a reliable strategy when subsystems interact significantly and may result in poor control performance. An apparent recourse is centralized control, which achieves optimal nominal control. Centralized control, however, is largely viewed by practitioners as inflexible and unsuitable for systemwide control. In plants operating with decentralized MPCs, conversion to centralized MPC control requires significant control system restructuring. Furthermore, managing a centralized MPC controller entails extensive model development, maintenance and data handling effort, which is undesirable.

A review of existing distributed MPC literature is available in the first part of this series [38]. All available distributed MPC formulations assume perfect knowledge of the states (state feedback) and do not address the case where the states of each subsystem are estimated from local measurements (output feedback). The states of a large, interacting system cannot usually be measured. Consequently, estimating the subsystem states from available measurements is a key component in any practical MPC implementation. The theory for centralized linear estimation is well understood. For large-scale systems, organizational and geographic constraints may preclude the use of centralized estimation strategies. The centralized Kalman filter requires measurements from all subsystems to estimate the state. For large, networked systems, the number of measurements is usually large to meet redundancy and robustness requirements. One difficulty with centralized estimation is communicating voluminous local measurement data to a central processor where the estimation algorithm is executed. Another difficulty is handling the vast amounts of data associated with centralized processing. Parallel solution techniques for estimation are available [16; 17]. While these techniques reduce the data transmission requirement, a central processor that updates the overall system error covariances at each time step is still necessary. Analogous to centralized control, the optimal, centralized estimator is a benchmark for evaluating the performance of different distributed estimation strategies. A decentralized estimator design framework for large-scale systems was proposed in [34; 35; 36]. Local estimators were designed based on the decentralized dynamics and additional compensatory inputs were included for each estimator to account for the interactions between the subsystems. Estimator convergence was established under assumptions on either the strength of the interconnections or the structure of the interconnection matrix. A decentralized estimator design strategy, in which the interconnections are treated as unknown inputs was proposed in [29; 40] for a restricted class of systems where the interconnections satisfy certain algebraic conditions and the number of outputs, for each subsystem, is greater than the number of interacting inputs.

Disturbance models are used to eliminate steady-state offset in the presence of nonzero mean, constant disturbances and/or plant-model mismatch. The output disturbance model is the most widely used disturbance model in industry to achieve zero offset control performance at steady state [7; 12; 27]. It is well known that output disturbance models cannot be used in plants with integrating modes as the effects of the augmented disturbance cannot be distinguished from the plant integrating modes. An alternative is to use input disturbance models [8], where the disturbances are assumed to enter the system through the inputs. For single (centralized) MPCs, [20; 23] derive conditions that guarantee zero offset control, using suitable disturbance models, in the presence of unmodelled effects and/or nonzero mean disturbances. In a distributed MPC framework, many choices for disturbance models exist. From a practitioner's standpoint, it is usually convenient to use local integrating disturbances. To track nonzero output setpoints, we require input and state targets that bring the system to the desired output targets at steady state. One option for determining the optimal steady-state targets in a distributed MPC framework is to perform a centralized target calculation [21] using the composite model for the plant. Alternatively, the target calculation problem can be formulated in a distributed manner with all the subsystem targets computed locally. A discussion on distributed target calculation is provided in Section 4.2.

Properties of the cooperative distributed MPC framework with distributed state estimation and distributed target calculation are investigated in this paper. The notation used in this paper conforms with the notation in [38]. In Section 2, a distributed estimation strategy is presented for estimating subsystem states from local measurements. A brief overview of the FC-MPC optimization problem and the associated algorithm is provided in Section 3. The distributed MPC control law under output feedback is defined and feasibility, optimality, and perturbed closed-loop stability are established for the proposed distributed MPC framework with state estimation. A disturbance modeling framework to achieve zero-offset steady-state control in the presence of nonzero mean disturbances and/or plant model mismatch is presented in Section 4. A distributed algorithm for computing the steady-state input, state and output targets is described subsequently. Conditions that ensure offset-free control at steady state are provided in Section 5. Two examples are presented in Section 6 to illustrate the efficacy of the proposed distributed MPC framework under output feedback. The main accomplishments of this paper are summarized in Section 7.

2 State estimation for FC-MPC

Lemma 1. *Detectability (and hence stabilizability) is invariant under a similarity transformation.*

The result follows from [4, Theorems 5.15 and 5.16, p. 200]. An alternate proof is given in Appendix A.

In the context of FC-MPC, the goal of state estimation is to determine the states of each subsystem (decentralized and interaction) from local measurements.

Assumption 1. All interaction models are stable *i.e.*, for each $i, j \in \mathbb{I}_M,\ |\lambda_{\max}(A_{ij})| < 1,\ \forall\, j \neq i$.

The following lemma provides a necessary and sufficient condition for detectability of each subsystem CM.

Lemma 2 (Detectability)**.** *Let Assumption 1 hold. The CM (A_i, C_i) is detectable if and only if (A_{ii}, C_{ii}) is detectable.*

A proof is given in Appendix A.

2.1 Distributed estimation with subsystem-based noise shaping matrices

We consider a distributed estimation framework in which the noise shaping matrix $G_i \in \mathbb{R}^{n_i \times g_i}$ and noise covariances Q_{x_i}, R_{v_i} are estimated locally for each $i \in \mathbb{I}_M$. The steady-state subsystem-based Kalman filters designed subsequently require only local measurements. For each subsystem $i \in \mathbb{I}_M$, let

$$x_i(k+1) = A_i x_i(k) + B_i u_i(k) + \sum_{j \neq i}^{M} W_{ij} u_j(k) + G_i w_{x_i}(k), \qquad w_{x_i}(k) \sim N(0, Q_{x_i}) \tag{1a}$$

$$y_i(k) = C_i x_i(k) + \nu_i(k), \qquad \nu_i(k) \sim N(0, R_{v_i}) \tag{1b}$$

denote the CM employed by each subsystem's Kalman filter, in which $w_{x_i} \sim N(0, Q_{x_i}) \in \mathbb{R}^{g_i}$ and $\nu_i \sim N(0, R_{v_i}) \in \mathbb{R}^{n_{y_i}}$ represent zero-mean white noise disturbances affecting the CM state equation and output equation, respectively.

Assumption 2. For each $i \in \mathbb{I}_M$, (A_{ii}, C_{ii}) is detectable.

Assumption 3. For each $i \in \mathbb{I}_M$, (A_{ii}, B_{ii}) is stabilizable.

Consider the CM $(A_i, B_i, \{W_{ij}\}_{j \neq i}, C_i)$ for subsystem $i \in \mathbb{I}_M$. Let Assumptions 2 and 3 be satisfied. Lemma 2 gives (A_i, C_i) is detectable and (A_i, B_i) is stabilizable. There exists a similarity transformation T_i that converts the CM for subsystem i into observability canonical form [14]. Let

$(\widehat{A}_i, \widehat{C}_i) = (T_i A_i T_i^{-1}, C_i T_i^{-1})$ be the A and C matrices of the CM in observability canonical form, where

$$\widehat{A}_i = \begin{bmatrix} A_i^{\mathrm{o}} & 0 \\ A_i^{12} & A_i^{\bar{\mathrm{o}}} \end{bmatrix}, \ \widehat{C}_i = \begin{bmatrix} C_i^{\mathrm{o}} & 0 \end{bmatrix} \tag{2}$$

From Lemma 1, $(\widehat{A}_i, \widehat{C}_i)$ is detectable. Therefore, $A_i^{\bar{\mathrm{o}}}$, which corresponds to the unobservable partition of the subsystem CM, is stable. The observable subsystem CM is

$$x_i^o(k+1) = A_i^{\mathrm{o}} x_i^o(k) + B_i^{\mathrm{o}} u_i(k) + \sum_{j \neq i} W_{ij}^{\mathrm{o}} u_j(k) + G_i^{\mathrm{o}} w_i(k), \quad y_i(k) = C_i^{\mathrm{o}} x_i^{\mathrm{o}}(k), \quad x_i^{\mathrm{o}} \in \mathbb{R}^{n_i^o}.$$

The noise covariances $G_i^{\mathrm{o}} Q_{x_i} G_i^{\mathrm{o}\prime}$ and R_{v_i} can be determined for the observable CM above, using any of autocovariance least squares (ALS) methods available in the literature [3; 18; 19; 22; 32]. Here, we use the procedure described in [22]. Since (A_i, C_i) is detectable, a stable estimator gain $\mathcal{L}_i$ exists and [22, Assumptions 1 and 2] are satisfied. The closed-loop data for estimating the covariances is generated using any stable filter gain for each estimator $i \in \mathbb{I}_M$. The FC-MPC algorithm (Algorithm 1, p. 7) is used for regulation. Two possible scenarios arise during estimation of the noise covariances. In the first case, $G_i^{\mathrm{o}} Q_{x_i} G_i^{\mathrm{o}\prime}$ and R_{v_i} can be estimated uniquely. A necessary and sufficient rank condition under which the ALS procedure gives unique estimates is given in [22, Lemma 4]. For the observable subsystem model $(A_i^{\mathrm{o}}, C_i^{\mathrm{o}})$ used in ALS estimation, unique estimates of $G_i^{\mathrm{o}} Q_{x_i} G_i^{\mathrm{o}\prime}$ and R_{v_i} are obtained only if $n_{y_i} \geq n_i^o$. For the case $n_{y_i} < n_i^o$, the estimates of $G_i^{\mathrm{o}} Q_{x_i} G_i^{\mathrm{o}\prime}$ and R_{v_i} are not unique. In this case, several choices for disturbance covariances that generate the same output data exist. One may choose any solution to the constrained ALS estimation problem [22, Equation 13, p. 307] to calculate the estimator gain. Let $\mathcal{G}_i^{\mathrm{o}} \mathcal{Q}_{x_i} \mathcal{G}_i^{\mathrm{o}\prime}$ and $\mathcal{R}_{v_i}$ represent a solution to the constrained ALS estimation problem of [22, Equation 13, p. 307] for subsystem $i \in \mathbb{I}_M$. A possible choice for the noise shaping matrix and the noise covariances is $G_i^{\mathrm{o}} \leftarrow I_{n_i}$, $Q_{x_i} \leftarrow \mathcal{G}_i^{\mathrm{o}} \mathcal{Q}_{x_i} \mathcal{G}_i^{\mathrm{o}\prime}$ and $R_{v_i} \leftarrow \mathcal{R}_{v_i}$. Another choice is $G_i^{\mathrm{o}} \leftarrow \mathcal{G}_i^{\mathrm{o}} \sqrt{\mathcal{Q}_{x_i}}$, $Q_{x_i} \leftarrow I_{g_i}$ and $R_{v_i} = \mathcal{R}_{v_i}$.

Lemma 3. *Let Assumptions 1 and 2 be satisfied. Define* $\widehat{G}_i = \begin{bmatrix} G_i^{\mathrm{o}} \\ 0 \end{bmatrix}$. $(\widehat{A}_i, \widehat{G}_i)$ *is stabilizable if and only if* $(A_i^{\mathrm{o}}, G_i^{\mathrm{o}})$ *is stabilizable.*

A proof is given in Appendix A.
Define $G_i = T_i^{-1} \widehat{G}_i$. From Lemma 1, (A_i, G_i) is stabilizable if and only if $(\widehat{A}_i, \widehat{G}_i)$ is stabilizable.

Corollary 1. *Let Assumptions 1 and 2 be satisfied. Let* $\widehat{G}_i = \begin{bmatrix} G_i^{\mathrm{o}} \\ 0 \end{bmatrix}$. $(\widehat{A}_i, \widehat{G}_i Q_{x_i}^{1/2})$ *is stabilizable if and only if* $(A_i^{\mathrm{o}}, G_i^{\mathrm{o}} Q_{x_i}^{1/2})$ *is stabilizable.*

Using the definition $G_i = T_i^{-1} \widehat{G}_i$, and Lemma 1, $(A_i, G_i Q_{x_i}^{1/2})$ is stabilizable if and only if $(\widehat{A}_i, \widehat{G}_i Q_{x_i}^{1/2})$ is stabilizable.

Remark 1. For each subsystem $i \in \mathbb{I}_M$, the conditions for the existence of a stable, steady-state Kalman filter are identical to those described in [24, Theorem 1] for a single (centralized) Kalman filter. Thus, if $R_{v_i} > 0$, $Q_{x_i} \geq 0$, (A_i, C_i) is detectable and $(A_i, G_i Q_{x_i}^{1/2})$ is stabilizable, the steady-state Kalman filter for subsystem i exists and is a stable estimator. If $Q_{x_i} > 0$, the requirements for stability of the steady-state Kalman filter reduce to $R_{v_i} > 0$, (A_i, C_i) is detectable and (A_i, G_i) is stabilizable [1].

Remark 2. The steady-state estimate error covariance for subsystem $i \in \mathbb{I}_M$, P_i, is the solution to the algebraic Riccati equation

$$P_i = G_i Q_{x_i} G_i{}' + A_i P_i A_i{}' - A_i P_i C_i{}' \left(R_{v_i} + C_i P_i C_i{}'\right)^{-1} C_i P_i A_i{}'.$$

The steady-state Kalman filter gain $\mathcal{L}_i$ for subsystem i is calculated as,

$$\mathcal{L}_i = P_i C_i{}' \left(R_{v_i} + C_i P_i C_i{}'\right)^{-1}.$$

Under the conditions of Remark 1, we have $|\lambda_{\max}(A_i - A_i \mathcal{L}_i C_i)| < 1$.

In this distributed estimation framework, the noise shaping matrix and noise covariances for each subsystem are identified using local process data. The estimators are decoupled, stable, and require only local measurement information. The estimates generated by each local Kalman filter may not be optimal, however. In [37, Chapter 5], an alternative strategy for distributed estimation, which allows interconnected noise shaping matrices, is presented. Details for this estimation strategy are omitted for brevity.

3 Closed-loop properties of FC-MPC under output feedback

A brief summary of the FC-MPC optimization problem and algorithm is provided. The reader is referred to [38, Sections 4 and 6] for details. It is assumed that the CM $(A_i, B_i, \{W_{ij}\}_{j\neq i}, C_i)$ is available for each subsystem $i \in \mathbb{I}_M$. The cost function for subsystem $i \in \mathbb{I}_M$ is

$$\phi_i\left(\mathbf{x}_i, \mathbf{u}_i; \widehat{x}_i\right) = \frac{1}{2}\sum_{l=0}^{\infty}\left[x_i(l)' Q_i x_i(l) + u_i(l)' R_i u_i(l)\right] \tag{3}$$

in which $x_i(0) = \widehat{x}_i, Q_i \geq 0$, $R_i > 0$ and $(A_i, Q_i^{1/2})$ is detectable. The cost function $\Phi_i\left([\mathbf{u}_1, \ldots, \mathbf{u}_M]; \widehat{x}_i\right)$ is obtained by eliminating $\mathbf{x}_i$, $i \in \mathbb{I}_M$ from $\phi_i(\cdot)$ (Equation (3)), using the appropriate CM (see [38, Equation (1)]. Define $\mathcal{U}_i = \Omega_i \times \ldots \times \Omega_i \in \mathbb{R}^{m_i N}$ for each $i \in \mathbb{I}_M$. The set of admissible subsystem inputs $\Omega_i, i \in \mathbb{I}_M$ is assumed to be a convex polytope (hence compact).The FC-MPC optimization problem for subsystem i, $\mathcal{F}_i$ is

$$\mathcal{F}_i \quad \triangleq \quad \min_{\overline{\mathbf{u}}_i} \quad \frac{1}{2}\overline{\mathbf{u}}_i{}'\mathcal{R}_i\overline{\mathbf{u}}_i \quad + \quad \left(r_i(k) + \sum_{j=1, j\neq i}^{M} \mathcal{H}_{ij}\overline{\mathbf{u}}_j^{p-1}(k)\right)' \overline{\mathbf{u}}_i(k) \quad + \quad \text{constant} \tag{4a}$$

subject to

$$\overline{\mathbf{u}}_i \in \mathcal{U}_i \tag{4b}$$

in which $\mathbb{Q}_i = \text{diag}(Q_i(1), \ldots, Q_i(N-1), \overline{Q}_i)$, $\mathbb{R}_i = \text{diag}(R_i(0), R_i(1), \ldots, R_i(N-1))$,

$$\mathcal{R}_i = w_i \mathbb{R}_i + w_i E_{ii}{}' \mathbb{Q}_i E_{ii} + \sum_{j\neq i}^{M} w_j E_{ji}{}' \mathbb{Q}_j E_{ji}, \qquad \mathcal{H}_{ij} = \sum_{l=1}^{M} w_l E_{li}{}' \mathbb{Q}_l E_{lj},$$

$$r_i(k) = w_i E_{ii}{}' \mathbb{Q}_i f_i \widehat{x}_i(k) + \sum_{j\neq i}^{M} w_j E_{ji}{}' \mathbb{Q}_j f_j \widehat{x}_j(k),$$

$$E_{ii} = \begin{bmatrix} B_i & 0 & \dots & \dots & 0 \\ A_iB_i & B_i & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_i^{N-1}B_i & \dots & \dots & \dots & B_i \end{bmatrix}, \ E_{ij} = \begin{bmatrix} W_{ij} & 0 & \dots & \dots & 0 \\ A_iW_{ij} & W_{ij} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_i^{N-1}W_{ij} & \dots & \dots & \dots & W_{ij} \end{bmatrix}, \ f_i = \begin{bmatrix} A_i \\ A_i^2 \\ \vdots \\ \vdots \\ A_i^N \end{bmatrix},$$

with $\overline{Q}_i$ denoting an appropriately chosen terminal penalty. The selection of this terminal penalty is described in the sequel. At each *iterate*, an optimization and exchange of information is performed for each subsystem. For subsystem i at iterate p, only the inputs $\mathbf{u}_i$ are optimized, updated and transmitted to the interconnected subsystems' MPCs. The input trajectories corresponding to the other subsystems are held constant at $\mathbf{u}_j^{p-1}, j \in \mathbb{I}_M, j \neq i$. For open-loop unstable systems, an additional terminal state constraint is necessary to ensure stability. Discussion on closed-loop stability is deferred until Section 3.

Assumption 4. $p_{\max} \in \mathbb{I}_+$ and $0 < p_{\max} < \infty$.

Assumption 5. $N \geq \max(\alpha, 1)$, in which $\alpha = \max(\alpha_1, \dots, \alpha_M)$ and $\alpha_i \geq 0$ denotes the number of unstable modes for subsystem $i \in \mathbb{I}_M$.

FC-MPC algorithm and properties

Algorithm 1 (FC-MPC). Given $\overline{\mathbf{u}}_i^0, \widehat{x}_i(k), \mathbb{Q}_i \geq 0, \mathbb{R}_i > 0, \ \forall\, i \in \mathbb{I}_M$
$p_{\max} > 0, \ \epsilon > 0, \ p \leftarrow 1, \kappa_i \leftarrow \Gamma\epsilon, \Gamma \gg 1$
while $\kappa_i > \epsilon$ for some $i \in \mathbb{I}_M$ and $p \leq p_{\max}$
  do $\forall \ i \in \mathbb{I}_M$
    $\overline{\mathbf{u}}_i^{*(p)} \in \ \arg(\mathcal{F}_i)$ (see Equation (4))
    $\overline{\mathbf{u}}_i^p \leftarrow w_i \overline{\mathbf{u}}_i^{*(p)} + (1 - w_i)\, \overline{\mathbf{u}}_i^{p-1}$
    Transmit $\overline{\mathbf{u}}_i^p$ to each interconnected subsystem $j \neq i$
    $\kappa_i \leftarrow \|\overline{\mathbf{u}}_i^p - \overline{\mathbf{u}}_i^{p-1}\|$
  end (do)
  $p \leftarrow p + 1$
end (while)

For the set of estimated states $\widehat{\mu}$, let $\Phi([\mathbf{u}_1^p, \dots, \mathbf{u}_M^p]; \widehat{\mu}) = \sum_{r=1}^M w_r \Phi_r([\mathbf{u}_1^p, \dots, \mathbf{u}_M^p]; \widehat{x}_r)$ represent the value of the cooperation-based cost function after p (Algorithm 1) iterates. In [38, Section 6], the following properties were established for the FC-MPC framework employing Algorithm 1.

Lemma 4. *Consider the FC-MPC formulation of Equation* (4). *The sequence of cost functions* $\left\{\Phi([\mathbf{u}_1^p, \mathbf{u}_2^p, \dots, \mathbf{u}_M^p]; \widehat{\mu})\right\}$ *generated by Algorithm 1 is a nonincreasing function of the iteration number* p.

See [38, Lemma 4, p. 17].

Lemma 5. *All limit points of Algorithm 1 are optimal.*

See [38, Lemma 5, p. 18].

3.1 Output feedback FC-MPC for distributed regulation

Assumption 6. $Q_i(0) = Q_i(1) = \ldots = Q_i(N-1) = Q_i > 0$ and $R_i(0) = R_i(1) = \cdots = R_i(N-1) = R_i > 0, \forall\, i \in \mathbb{I}_M$.

Assumption 7. For each $i \in \mathbb{I}_M$, $(A_i - A_i\mathcal{L}_iC_i)$ is stable.

The evolution of the estimate error is given by $e_i(k+1) = (A_i - A_i\mathcal{L}_iC_i)e_i(k)$, in which $e_i(k)$ is the state estimate error for subsystem $i \in \mathbb{I}_M$ at time k. From Assumption 7 and Equation (19) (see Appendix B.1), $Z_i = \mathcal{L}_iC_i(A_i - A_i\mathcal{L}_iC_i), i \in \mathbb{I}_M$.

FC-MPC control law under output feedback At time k and set of estimated subsystem states $\widehat{\mu}(k)$, let the FC-MPC algorithm (Algorithm 1) be terminated after $p(k) = q \geq 1$ cooperation-based iterates. Let

$$\mathbf{u}_i^q(\widehat{\mu}(k)) = \left[u_i^q(\widehat{\mu}(k),0)', u_i^q(\widehat{\mu}(k),1)', \ldots, u_i^q(\widehat{\mu}(k),N-1)',\ 0,\ 0, \ldots\right]',\ \forall\, i \in \mathbb{I}_M$$

represent the solution to Algorithm 1 after q iterates. The input injected into subsystem $i \in \mathbb{I}_M$ is $u_i^q(\widehat{\mu}(k),0)$. Let

$$\mathbf{u}_i^+(\widehat{\mu}(k)) = [u_i^p(\widehat{\mu}(k),1)', \ldots, u_i^p(\widehat{\mu}(k),N-1)', 0, 0, \ldots]' \tag{5}$$

represent a shifted version of $\mathbf{u}_i^p(\widehat{\mu}(k)), i \in \mathbb{I}_M$.

3.1.1 Stable decentralized modes

Initialization. At time 0, each MPC is initialized with the zero input trajectory $u_i(j|0) = 0,\ 0 \leq j,\ \forall\, i \in \mathbb{I}_M$. At time $k+1$, the initial input trajectory for each subsystem's MPC is $\mathbf{u}_i^0(k+1) = \mathbf{u}_i^+(\widehat{\mu}(k)),\ i \in \mathbb{I}_M$. The cost function value for the set of feasible initial subsystem input trajectories at $k+1$ is $J_N^0(\widehat{\mu}(k+1)) = \Phi\left([\mathbf{u}_1^0(k+1), \mathbf{u}_2^0(k+1), \ldots, \mathbf{u}_M^0(k+1)]; \widehat{\mu}(k+1)\right)$.

Feasibility and domain of attraction. For $(\widehat{x}_{ii}(0), e_i(0)) \in \mathbb{R}^{n_{ii}} \times \mathbb{R}^{n_i}$, the zero input trajectory is feasible for each $i \in \mathbb{I}_M$. Existence of a feasible input trajectory for each $i \in \mathbb{I}_M$ at $k = 0$ and $p(0) = 0$ guarantees feasibility of $\mathcal{F}_i,\ \forall\, i \in \mathbb{I}_M$ (Equation (4)) at all $k \geq 0$, $p(k) > 0$. This result follows from the initialization procedure, convexity of $\Omega_i, \forall\, i \in \mathbb{I}_M$ and Algorithm 1. The controllable domain for the nominal closed-loop system is $\mathbb{R}^n \times \mathbb{R}^n$.

Assumption 8. For each $i \in \mathbb{I}_M$, A_{ii} is stable, $\mathbb{Q}_i = \text{diag}\left(Q_i(1), \ldots, Q_i(N-1), \overline{Q}_i\right)$, in which $\overline{Q}_i$ is the solution of the Lyapunov equation $A_i{}'\overline{Q}_iA_i - \overline{Q}_i = -Q_i$

Exponential stability for the closed-loop system under the output feedback distributed MPC control law is stated in the following theorem, which requires that the local estimators are exponentially stable but makes no assumptions on the optimality of the estimates.

Theorem 1 (Stable modes)**.** *Consider Algorithm 1 employing the FC-MPC optimization problem of Equation (4). Let Assumptions 1 to 8 hold. The origin is an exponentially stable equilibrium for the perturbed closed-loop system*

$$\begin{aligned}
\widehat{x}_i(k+1) &= A_i\widehat{x}_i(k) + B_iu_i^p(\widehat{\mu}(k),0) + \sum_{j\neq i} W_{ij}u_j^p(\widehat{\mu}(k),0) + Z_ie_i, \\
e_i(k+1) &= (A_i - A_i\mathcal{L}_iC_i)e_i(k), \qquad i \in \mathbb{I}_M,
\end{aligned}$$

for all $((\widehat{x}_i(0), e_i(0)),\ i \in \mathbb{I}_M) \in \mathbb{R}^n \times \mathbb{R}^n$ *and all* $p = 1, 2, \ldots, p_{\max}$.

The proof is given in Appendix B.

3.1.2 Unstable modes

For systems with unstable modes, a terminal state constraint that forces the unstable modes to the origin at the end of the control horizon is employed in each FC-MPC optimization problem (Equation (4)). This terminal state constraint is necessary for stability. From Assumption 1, unstable modes, if any, are present only in the decentralized model. We have, therefore, that ${U_{u_i}}'\widehat{x}_i = {S_{u_i}}'\widehat{x}_{ii}, i \in \mathbb{I}_M$, in which U_{u_i} and S_{u_i}, are obtained through a Schur decomposition of A_i and A_{ii} respectively [1]. For $i \in \mathbb{I}_M$, define,

$$\mathbb{S}_i = \{x_{ii} \mid \exists\, \overline{\mathbf{u}}_i \in \mathcal{U}_i \text{ such that } {S_{u_i}}'[\mathcal{C}_N(A_{ii}, B_{ii})\, \overline{\mathbf{u}}_i + A_{ii}^N x_{ii}] = 0\} \qquad \text{steerable set}$$

to be the set of unstable decentralized modes that can be steered to zero in N moves. For stable systems, $\mathbb{S}_i = \mathbb{R}^{n_{ii}},\ i \in \mathbb{I}_M$. By definition, ${U_{u_i}}'[\mathcal{C}_N(A_i, B_i)\, \overline{\mathbf{u}}_i + A_i^N \widehat{x}_i] = {S_{u_i}}'[\mathcal{C}_N(A_{ii}, B_{ii})\, \overline{\mathbf{u}}_i + A_{ii}^N \widehat{x}_{ii}]$. From Assumption 1 and because the domain of each $x_{ij}, i, j \in \mathbb{I}_M, j \neq i$ is $\mathbb{R}^{n_{ij}}$,

$$\mathbb{D}_{R_i} = \mathbb{R}^{n_{i1}} \times \cdots \times \mathbb{R}^{n_{i(i-1)}} \times \mathbb{S}_i \times \mathbb{R}^{n_{i(i+1)}} \times \cdots \times \mathbb{R}^{n_{iM}} \subseteq \mathbb{R}^{n_i},\ i \in \mathbb{I}_M \qquad \text{domain of regulator}$$

represents the set of all x_i for which an admissible input trajectory $\overline{\mathbf{u}}_i$ exists that drives the unstable decentralized modes ${U_{u_i}}'x_i$ to the origin. A positively invariant set, $\mathbb{D}_C$, for the perturbed closed-loop system

$$\widehat{x}_i^+ = A_i \widehat{x}_i + B_i u_i^p(\widehat{\mu}, 0) + \sum_{j \neq i}^{M} W_{ij} u_j^p(\widehat{\mu}, 0) + Z_i e_i, \quad e_i^+ = (A_i - A_i \mathcal{L}_i C_i) e_i, \quad i \in \mathbb{I}_M \tag{6}$$

is given by

$$\mathbb{D}_C = \{((\widehat{x}_i, e_i),\ i \in \mathbb{I}_M) \mid \big((\widehat{x}_i^+, e_i^+),\ i \in \mathbb{I}_M\big) \in \mathbb{D}_C,\ \widehat{x}_i \in \mathbb{D}_{R_i}, i \in \mathbb{I}_M\} \qquad \text{domain of controller} \tag{7}$$

The set $\mathbb{D}_C$ can be constructed for the system described by Equation (6) using any of the techniques available in the literature for backward construction of polytopic sets under state and control constraints [2; 13; 15; 26].

Initialization. At time 0, let $((\widehat{x}_i(0), e_i(0)),\ i \in \mathbb{I}_M) \in \mathbb{D}_C$. A feasible input trajectory, therefore, exists for each $i \in \mathbb{I}_M$, and can be computed by solving the following quadratic program (QP).

$$\overline{\mathbf{u}}_i^0(0) = \arg\min_{\overline{\mathbf{u}}_i} \quad \|\overline{\mathbf{u}}_i\|^2$$
$$\text{subject to}$$
$$\overline{\mathbf{u}}_i \in \mathcal{U}_i$$
$${U_{u_i}}'[\mathcal{C}_N(A_i, B_i)\, \overline{\mathbf{u}}_i + A_i^N \widehat{x}_i(0)] = 0$$

---

[1] The Schur decomposition of $A_{ii} = \begin{bmatrix} S_{s_i} & S_{u_i} \end{bmatrix} \begin{bmatrix} A_{s_{ii}} & \bigoplus \\ & A_{u_{ii}} \end{bmatrix} \begin{bmatrix} {S_{s_i}}' \\ {S_{u_i}}' \end{bmatrix}$, $A_i = \begin{bmatrix} U_{s_i} & U_{u_i} \end{bmatrix} \begin{bmatrix} A_{s_i} & \bigotimes \\ & A_{u_i} \end{bmatrix} \begin{bmatrix} {U_{s_i}}' \\ {U_{u_i}}' \end{bmatrix}$. Eigenvalues of $A_{u_{ii}}, A_{u_i}$ are on or outside the unit circle. Eigenvalues of $A_{s_{ii}}, A_{s_i}$ are strictly inside the unit circle.

We have $\mathbf{u}_i^0(0) = [\overline{\mathbf{u}}_i^0(0), 0, 0, \ldots\ldots]$. Let $\overline{\mathbf{u}}_i^+(\widehat{\mu}(k)) = [u_i^{p(k)}(\widehat{\mu}(k), 1)', \ldots, u_i^{p(k)}(\widehat{\mu}(k), N-1)', 0]'$ represent a shifted version of $\overline{\mathbf{u}}_i^{p(k)}(\widehat{\mu}(k))$. For times $k > 0$, $\overline{\mathbf{u}}_i^0(k) = \overline{\mathbf{u}}_i^+(\widehat{\mu}(k-1)) + \overline{\mathbf{v}}_i(k)$, in which $\overline{\mathbf{v}}_i(k) \in \mathbb{R}^{m_i N}$ is calculated by solving the following QP for each $i \in \mathbb{I}_M$.

$$\overline{\mathbf{v}}_i(k) = \arg\min_{\overline{\mathbf{v}}_i(k)} \quad \|\overline{\mathbf{v}}_i(k)\|^2 \tag{8a}$$

subject to

$$\overline{\mathbf{u}}_i^+(\widehat{\mu}(k-1)) + \overline{\mathbf{v}}_i(k) \in \mathcal{U}_i \tag{8b}$$

$$U_{u_i}'[\mathcal{C}_N(A_i, B_i)\ \left(\overline{\mathbf{u}}_i^+(\widehat{\mu}(k-1), 1) + \overline{\mathbf{v}}_i(k)\right) + A_i^N \widehat{x}_i(k)] = 0 \tag{8c}$$

Since $\mathbb{D}_C$ is positively invariant (by construction) and $((\widehat{x}_i(0), e_i(0)),\ i \in \mathbb{I}_M) \in \mathbb{D}_C$, a solution to the optimization problem of Equation (8) exists for all $i \in \mathbb{I}_M$ and all $k > 0$. Define $\mathbf{v}_i(k) = [\overline{\mathbf{v}}_i(k)', 0, 0, \ldots]'$.

Assumption 9. $\alpha > 0$ (see Assumption 5). For each $i \in \mathbb{I}_M$, $\mathbb{Q}_i = \text{diag}\left(Q_i(1), \ldots, Q_i(N-1), \overline{Q}_i\right)$, in which $\overline{Q}_i = U_{s_i}\Sigma_i U_{s_i}'$ with Σ_i obtained as the solution of the Lyapunov equation $A_{s_i}'\Sigma_i A_{s_i} - \Sigma_i = -U_{s_i}'Q_i U_{s_i}$.

The following theorem establishes exponential closed-loop stability under output feedback for systems with unstable modes.

Theorem 2 (Unstable modes). *Let Assumptions 1 to 7 and Assumption 9 hold. Consider Algorithm 1, using the FC-MPC optimization problem of Equation (4) with an additional end constraint* $U_{u_i}'\widehat{x}_i(k+N|k) = U_{u_i}'[\mathcal{C}_N(A_i, B_i)\,\overline{\mathbf{u}}_i + A_i^N \widehat{x}_i(k)] = 0$ *enforced on the unstable decentralized modes. The origin is an exponentially stable equilibrium for the perturbed closed-loop system*

$$\begin{aligned}
\widehat{x}_i(k+1) &= A_i\widehat{x}_i(k) + B_i u_i^p(\widehat{\mu}(k), 0) + \sum_{j \neq i} W_{ij} u_j^p(\widehat{\mu}(k), 0) + Z_i e_i, \\
e_i(k+1) &= (A_i - A_i\mathcal{L}_i C_i)e_i(k), \qquad i \in \mathbb{I}_M,
\end{aligned}$$

for all $((\widehat{x}_i(0), e_i(0)),\ i \in \mathbb{I}_M) \in \mathbb{D}_C$ *(Equation* (7)*) and all* $p = 1, 2, \ldots, p_{\max}$.

The proof is given in Appendix B.

Remark 3. In [38], it is shown that the nominal distributed MPC control law is exponentially stable. Lipschitz continuity of the nominal distributed MPC control law in the subsystem states μ is established. Asymptotic stability of the output feedback distributed MPC control law under decaying perturbations follows using [31, Theorem 3].

4 Disturbance modeling and distributed target calculation for FC-MPC

4.1 Disturbance modeling for FC-MPC

For each subsystem $i \in \mathbb{I}_M$, the CM state is augmented with the integrating disturbance. The augmented CM $(\widetilde{A}_i, \widetilde{B}_i, \{\widetilde{W}_{ij}\}_{j\neq i}, \widetilde{C}_i, \widetilde{G}_i)$ for subsystem i is

$$\underbrace{\begin{bmatrix} x_i \\ d_i \end{bmatrix}}_{\widetilde{x}_i}(k+1) = \underbrace{\begin{pmatrix} A_i & B_i^d \\ 0 & I \end{pmatrix}}_{\widetilde{A}_i} \begin{bmatrix} x_i \\ d_i \end{bmatrix}(k) + \underbrace{\begin{pmatrix} B_i \\ 0 \end{pmatrix}}_{\widetilde{B}_i} u_i(k) + \sum_{j\neq i}^{M} \underbrace{\begin{pmatrix} W_{ij} \\ 0 \end{pmatrix}}_{\widetilde{W}_{ij}} u_j(k) + \underbrace{\begin{pmatrix} G_i & \\ & I \end{pmatrix}}_{\widetilde{G}_i} \begin{bmatrix} w_{x_i} \\ w_{d_i} \end{bmatrix}(k),$$

$$y_i(k) = \underbrace{\begin{pmatrix} C_i & C_i^d \end{pmatrix}}_{\widetilde{C}_i} \begin{bmatrix} x_i \\ d_i \end{bmatrix}(k) + \nu_i(k),$$

in which $d_i \in \mathbb{R}^{n_{d_i}}$, $B_i^d \in \mathbb{R}^{n_i \times n_{d_i}}$, $C_i^d \in \mathbb{R}^{n_{y_i} \times n_{d_i}}$. The vectors $w_{x_i}(k) \sim N(0, Q_{x_i}) \in \mathbb{R}^{n_{g_i}}$, $w_{d_i}(k) \sim N(0, Q_{d_i}) \in \mathbb{R}^{n_{d_i}}$ and $\nu_i(k) \sim N(0, R_{v_i}) \in \mathbb{R}^{n_{y_i}}$ are zero mean white noise disturbances affecting the augmented CM state equation and output equation, respectively. The notation (B_i^d, C_i^d) represents the input–output disturbance model pair for subsystem i, in which $B_i^d = \mathrm{vec}(B_{i1}^d, \ldots, B_{ii}^d, \ldots, B_{iM}^d)$. The augmented decentralized model $(\widetilde{A}_{ii}, \widetilde{B}_{ii}, \widetilde{C}_{ii})$ is obtained by augmenting the decentralized state x_{ii} with the integrating disturbance d_i. It is assumed that the augmented decentralized model with the input-output disturbance model pair $(\mathcal{B}_{ii}^d, \mathcal{C}_i^d)$ is detectable (hence, $n_{d_i} \leq n_{y_i}$) [2].

Lemma 6. *Let Assumptions 1 and 2 hold. For each subsystem $i \in \mathbb{I}_M$, let the augmented decentralized model $(\widetilde{A}_{ii}, \widetilde{C}_{ii})$ with the input-output disturbance model pair $(\mathcal{B}_{ii}^d, \mathcal{C}_i^d)$ be detectable. The augmented CM $(\widetilde{A}_i, \widetilde{C}_i)$ with input disturbance model $B_i^d = \mathrm{vec}(B_{i1}^d, \ldots B_{ii}^d, \ldots, B_{iM}^d)$, in which $B_{ii}^d = \mathcal{B}_{ii}^d$, $B_{ij}^d = 0,\ j \in \mathbb{I}_M,\ j \neq i$, and output disturbance model $\mathcal{C}_i^d$, is detectable.*

A proof is given in Appendix C.

In view of the internal model principle [9], it may be preferable to choose disturbance models that best represent the actual plant disturbances. Hence, in certain cases, it may be useful to use a more general input disturbance model of the form $B_i^d = \mathrm{vec}(B_{i1}^d, \ldots, B_{ii}^d, \ldots, B_{iM}^d)$ in conjunction with the output disturbance model C_i^d. The following lemma gives a general condition for detectability of the augmented CM $(\widetilde{A}_i, \widetilde{C}_i)$.

Lemma 7. *Let Assumption 1 and 2 hold. The augmented CM $(\widetilde{A}_i, \widetilde{C}_i)$, with input disturbance model $B_i^d = \mathrm{vec}(B_{i1}^d, \ldots, B_{ii}^d, \ldots, B_{iM}^d)$ and output disturbance model C_i^d, is detectable if and only if*

$$\mathrm{rank} \begin{bmatrix} I - A_i & -B_i^d \\ C_i & C_i^d \end{bmatrix} = n_i + n_{d_i} \tag{9}$$

A proof is presented in Appendix C.

One method to satisfy the rank condition of Equation (9) is to pick B_i^d and C_i^d such that $\mathrm{range}\left(\begin{bmatrix} B_i^d \\ -C_i^d \end{bmatrix}\right) \subseteq \mathrm{null}\left(\begin{bmatrix} I - A_i \\ C_i \end{bmatrix}'\right)$. Let $(\widetilde{A}_i, \widetilde{C}_i)$ be detectable and let the steady-state estimator

[2] Conditions for detectability of the augmented decentralized model are given in [23, Lemma 1, p. 431]

gain for the state and integrating disturbance vector for subsystem i be denoted by $\mathcal{L}_{\mathrm{x}_i}$ and $\mathcal{L}_{\mathrm{d}_i}$ respectively. The filter equations for subsystem i are

$$\begin{bmatrix} \widehat{x}_i \\ \widehat{d}_i \end{bmatrix}(k|k) = \begin{bmatrix} \widehat{x}_i \\ \widehat{d}_i \end{bmatrix}(k|k-1) + \begin{bmatrix} \mathcal{L}_{\mathrm{x}_i} \\ \mathcal{L}_{\mathrm{d}_i} \end{bmatrix} \Big(y_i(k) - C_i \widehat{x}_i(k|k-1) - C_i^d \widehat{d}_i(k|k-1) \Big) \tag{10a}$$

$$\begin{bmatrix} \widehat{x}_i \\ \widehat{d}_i \end{bmatrix}(k+1|k) = \begin{bmatrix} A_i & B_i^d \\ & I \end{bmatrix} \begin{bmatrix} \widehat{x}_i \\ \widehat{d}_i \end{bmatrix}(k|k) + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i(k) + \sum_{j\neq i}^{M} \begin{bmatrix} W_{ij} \\ 0 \end{bmatrix} u_j(k) \tag{10b}$$

in which $\widehat{x}_i, \widehat{d}_i$ denotes an estimate of the state and integrating disturbance respectively, for subsystem i.

4.2 Distributed target calculation

For robustness and redundancy, the number of measurements is typically chosen greater than the number of manipulated inputs. Consequently, one can achieve offset-free control for only a subset of the measured variables. Define $z_i = H_i y_i, z_i \in \mathbb{R}^{n_{c_i}}, H_i \in \mathbb{R}^{n_{c_i} \times n_{y_i}}$ to be the set of controlled variables (CVs) for each subsystem $i \in \mathbb{I}_M$. The choice of CVs is presumed to satisfy:

Assumption 10.

$$\operatorname{rank} \begin{bmatrix} I - A_{ii} & -B_{ii} \\ H_i C_{ii} & 0 \end{bmatrix} = n_{ii} + n_{c_i},\ i \in \mathbb{I}_M, \tag{11}$$

Assumption 10 implies that the number of CVs for each subsystem $i \in \mathbb{I}_M$ cannot exceed either the number of manipulated variables (MVs) m_i or the number of measurements n_{y_i}, and that $H_i C_{ii}$ must be full row rank.

Lemma 8. *Let Assumptions 1 and 10 hold.*

$$\operatorname{rank} \begin{bmatrix} I - A_i \\ H_i C_i \end{bmatrix} = n_i \ \textit{ if and only if } \ \operatorname{rank} \begin{bmatrix} I - A_{ii} \\ H_i C_{ii} \end{bmatrix} = n_{ii}.$$

The proof is similar to the proof for Lemma 2, and is omitted. □

Assumption 11. $\operatorname{rank} \begin{bmatrix} I - A_{ii} \\ H_i C_{ii} \end{bmatrix} = n_{ii},\ i \in \mathbb{I}_M$.

Assumption 11 is a weaker restriction than detectability of $(A_{ii}, H_i C_{ii})$. In the distributed target calculation framework, the steady-state targets are computed at the subsystem level. At each iterate, an optimization and exchange of calculated steady-state information among interacting subsystems is performed. For subsystem $i \in \mathbb{I}_M$, let z_i^{sp} denote the setpoint for the CVs and let u_i^{ss} represent the corresponding steady-state value for the MVs. Hence, we write $z_i^{\mathrm{sp}} = \mathcal{G}_i u_i^{\mathrm{ss}}$, where $\mathcal{G}_i$ is a steady-state gain matrix. The triplet $(y_{s_i}, x_{s_i}, u_{s_i})$ represents the steady-state output, state and input target for subsystem i. The target objective for subsystem $i \in \mathbb{I}_M$, Ψ_i, is defined as $\Psi_i(u_{s_i}) = \frac{1}{2}(u_i^{\text{ss}} - u_{s_i})' R_{u_i}(u_i^{\text{ss}} - u_{s_i})$, in which $R_{u_i} > 0$. Each subsystem $i \in \mathbb{I}_M$, solves the following QP at iterate t.

$$(x_{s_{ii}}^{*(t)}, u_{s_i}^{*(t)}) \in \arg\min_{x_{s_{ii}}, u_{s_i}} \frac{1}{2}(u_i^{\text{ss}} - u_{s_i})' R_{u_i}(u_i^{\text{ss}} - u_{s_i}) \tag{12a}$$

subject to

$$u_{s_i} \in \Omega_i \tag{12b}$$

$$\begin{bmatrix} I - A_{ii} & -B_{ii} \\ H_i C_{ii} & \end{bmatrix} \begin{bmatrix} x_{s_{ii}} \\ u_{s_i} \end{bmatrix} = \begin{bmatrix} B_{ii}^d \widehat{d}_i \\ z_i^{\text{sp}} - H_i C_i^d \widehat{d}_i - \sum_{j \neq i}^M \left(\overline{g}_{ij} u_{s_j}^{t-1} + \overline{h}_{ij} \widehat{d}_i \right) \end{bmatrix} \tag{12c}$$

in which $\overline{g}_{ij} = H_i C_{ij}(I - A_{ij})^{-1} B_{ij}$, $\overline{h}_{ij} = H_i C_{ij}(I - A_{ij})^{-1} B_{ij}^d$, $\forall\, i, j \in \mathbb{I}_M,\ j \neq i$.

Existence. Let Assumption 10 be satisfied. Consider

$$\begin{bmatrix} I - A_{ii} & B_{ii} \\ H_i C_{ii} & \end{bmatrix} \begin{pmatrix} x_{s_{ii}} \\ u_{s_i} \end{pmatrix} = \begin{bmatrix} B_{ii}^d \widehat{d}_i \\ z_i^{\text{sp}} - H_i C_i^d \widehat{d}_i - \sum_{j \neq i}^M \left(\overline{g}_{ij} u_{s_j} + \overline{h}_{ij} \widehat{d}_i \right) \end{bmatrix},\ i \in \mathbb{I}_M \tag{13}$$

$$\mathbb{D}_T = \Big\{ \Big((z_i^{\text{sp}}, \widehat{d}_i),\ i \in \mathbb{I}_M \Big) \mid \exists\ ((x_{s_{ii}}, u_{s_i}), i \in \mathbb{I}_M) \text{ satisfying Equation (13) and } u_{s_j} \in \Omega_j,\ \forall\, j \in \mathbb{I}_M \Big\} \qquad \text{domain of target}$$

If $\mathbb{D}_T$ is empty, the constraints are too stringent to meet z_i^{sp}, $i \in \mathbb{I}_M$. For $\mathbb{D}_T$ nonempty and $\Big((z_i^{\text{sp}}, \widehat{d}_i), i \in \mathbb{I}_M \Big) \in \mathbb{D}_T$, the feasible region for Equation (12) is nonempty for each $i \in \mathbb{I}_M$. Since $R_{u_i} > 0$, the objective is bounded below. A solution to Equation (12), therefore, exists for all $i \in \mathbb{I}_M$ [10].

Uniqueness.

Lemma 9. *For each subsystem $i \in \mathbb{I}_M$, let H_i satisfy Assumption 10. Let Assumption 1 hold. The solution to the target optimization problem (Equation* (12)*) for each $i \in \mathbb{I}_M$, if it exists, is unique if and only if Assumption 11 is satisfied.*

A proof is given in Appendix C. □

Corollary 2. $x_{s_i}^{*(t)} = [x_{s_{i1}}^{*(t)}{}', \ldots, x_{s_{iM}}^{*(t)}{}']', i \in \mathbb{I}_M$ *is unique.*

Remark 4. It can be shown that $(A_i, H_i C_i)$ is detectable if and only if $(A_{ii}, H_i C_{ii})$ is detectable. For subsystem i, if H_i satisfies Assumption 10, $(A_{ii}, H_i C_{ii})$ is detectable and Assumption 1 holds, the solution to the optimization problem of Equation (12) is unique.

The steady-state targets for each $i \in \mathbb{I}_M$ are obtained using the distributed target calculation algorithm given below.

Algorithm 2. Given $\left(u_{s_i}^0,\, z_i^{\text{sp}},\, u_i^{\text{ss}}\right),\ R_{u_i} > 0,\, \forall\, i \in \mathbb{I}_M, t_{\max} > 0,\ \epsilon > 0$
$t \leftarrow 1,\ \kappa_i \leftarrow \Gamma\epsilon,\ \Gamma \gg 1$
while $\kappa_i > \epsilon$ for some $i \in \mathbb{I}_M$ and $t \leq t_{\max}$
  do $\forall\, i \in \mathbb{I}_M$
    Determine $(x_{s_{ii}}^{*(t)}, u_{s_i}^{*(t)})$ from Equation (12)
    $(x_{s_{ii}}^t, u_{s_i}^t) \leftarrow w_i(x_{s_{ii}}^{*(t)}, u_{s_i}^{*(t)}) + (1-w_i)(x_{s_{ii}}^{t-1}, u_{s_i}^{t-1})$
    $\kappa_i \leftarrow \|(x_{s_{ii}}^t, u_{s_i}^t) - (x_{s_{ii}}^{t-1}, u_{s_i}^{t-1})\|$
    Transmit $(x_{s_{ii}}^t, u_{s_i}^t)$ to each interconnected subsystem $j \in \mathbb{I}_M,\ j \neq i$
  end (do)
$t \leftarrow t+1$
end (while)

For each subsystem $i \in \mathbb{I}_M$ at iterate t, the target state $x_{s_{ij}}^t, j \neq i$ is calculated using $x_{s_{ij}}^t = (I - A_{ij})^{-1}\left[B_{ij}u_{s_j}^t + B_{ij}^d\widehat{d}_i\right]$ and by definition, $x_{s_i}^t = [{x_{s_{i1}}^t}', \ldots, {x_{s_{iM}}^t}']'$. All iterates generated by Algorithm 2 are feasible steady states. Furthermore, the target calculation objective $\Psi(\cdot) = \sum_{i=1}^M w_i\Psi_i(u_{s_i}^t)$, in which $w_i > 0,\ i \in \mathbb{I}_M$ and $\sum_{i=1}^M w_i = 1$, is a nonincreasing function of the iteration number t. Since $\Psi_i(\cdot), i \in \mathbb{I}_M$ is bounded below, the sequence of costs $\{\Psi(u_{s_1}^t, \ldots, u_{s_M}^t)\}$ converges. The proof for monotonicity and convergence is identical to the proof for Lemma 4 and is omitted. Let $\Psi(u_{s_1}^t, \ldots, u_{s_M}^t) \to \Psi^\infty$. Because an equality constraint that couples input targets of different subsystems is included in the target optimization (Equation (12)), optimality at convergence may not apply for Algorithm 2. Define the limit set $S^\infty = \{((x_{s_{ii}}, u_{s_i}),\ i \in \mathbb{I}_M) \mid \Psi(\cdot) = \Psi^\infty\}$. It can be shown that the sequence $(x_{s_{ii}}^t, u_{s_i}^t),\ i \in \mathbb{I}_M$ (generated by Algorithm 2) converges to a point $\left((x_{s_{ii}}^\infty, u_{s_i}^\infty),\ i \in \mathbb{I}_M\right) \in S^\infty$. The targets $(x_{s_{ii}}^\infty, u_{s_i}^\infty),\ i \in \mathbb{I}_M$ may be different from $(x_{s_i}^*, u_{s_i}^*),\ i \in \mathbb{I}_M$, the optimal state and input targets obtained using a centralized target calculation [37]. However, $z_i^{\text{sp}} = C_i x_{s_i}^\infty + C_d^d\widehat{d}_i$ and $(I - A_i)x_{s_i}^\infty = B_i u_{s_i}^\infty + \sum_{j\neq i}^M W_{ij}u_{s_j}^\infty + B_i^d\widehat{d}_i,\ i \in \mathbb{I}_M$.

4.3 Initialization

At time $k = 0$, Algorithm 2 is initialized with any feasible $(x_{s_{ii}}^0(0), u_{s_i}^0(0)),\ i \in \mathbb{I}_M$. Let Algorithm 2 be terminated after $t \in \mathbb{I}_+, t \leq t_{\max}$ iterates. For the nominal or constant disturbance case, the steady-state pair $(x_{s_i}^t(k), u_{s_i}^t(k)),\ i \in \mathbb{I}_M$ is a feasible initial guess for the distributed target optimization problem (Equation (12)) at time $k+1$. Using monotonicity and convergence properties for Algorithm 2, $(x_{s_i}^t(k), u_{s_i}^t(k))$ converges to $(x_{s_i}^\infty, u_{s_i}^\infty),\ i \in \mathbb{I}_M$ as $k \to \infty$.

5 Offset-free control with FC-MPC

The regulation problem of Section 3 assumed the input and output targets are at the origin. The targets may of course need to be nonzero while tracking nonzero setpoints or rejecting nonzero constant disturbances. To achieve offset-free control in the above scenarios, a target calculation is performed and the target shifted states, inputs and outputs are used in the regulator. In the FC-MPC framework, Algorithm 2 may be terminated at intermediate iterates. For a large, networked system, the number of local measurements usually exceeds the number of subsystem inputs. Also, only the CVs typically have setpoints. Offset-free control can, therefore, be achieved for only the CVs. The choice of regulator parameters is restricted to enable offset-free control in the CVs.

Accordingly, the stage cost for subsystem i is defined as

$$L_i(\widehat{\omega}_i, \nu_i) = \frac{1}{2}\left[\|z_i - z_i^{\text{sp}}\|^2_{Q_{z_i}} + \|u_i - u^t_{s_i}\|^2_{R_i}\right] = \frac{1}{2}\left[\widehat{\omega}_i{}' Q_i \widehat{\omega}_i + \nu_i{}' R_i \nu_i\right]$$

in which t is the number of distributed target calculation iterates, $\widehat{\omega}_i = \widehat{x}_i - x^t_{s_i}$, $\nu_i = u_i - u^t_{s_i}$, $z_i^{\text{sp}} = H_i C_i x^t_{s_i} + H_i C^d_i \widehat{d}_i$, $Q_{z_i}, R_i > 0$, $Q_i = C_i{}' H_i{}' Q_{z_i} H_i C_i$ and $\left((z_i^{\text{sp}}, \widehat{d}_i),\ i \in \mathbb{I}_M\right) \in \mathbb{D}_T$. The cost function for subsystem i (Equation (3)) is rewritten as $\phi_i(\widehat{\boldsymbol{\omega}}_i(k), \boldsymbol{\nu}_i(k); \widehat{\omega}_i(k))$, where $\widehat{\boldsymbol{\omega}}_i(k) = [\omega_i(k+1|k)', \omega_i(k+2|k)', \ldots]'$, $\boldsymbol{\nu}_i(k) = [\nu_i(k)', \nu_i(k+1|k), \ldots]$ and $\widehat{\omega}_i(k+j+1|k) = A_i \widehat{\omega}_i(k+j|k) + B_i \nu_i(k+j|k) + \sum_{s \neq i}^{M} W_{is} \nu_s(k+j|k),\ 0 \leq j$. It can be shown under the assumption $Q_{z_i} > 0$ that $(A_i, Q_i^{1/2})$ (with Q_i defined as above) is detectable if and only if $(A_{ii}, H_i C_{ii})$ is detectable. Let $\overline{\boldsymbol{\nu}}_i(k) = [\nu_i(k)', \ldots, \nu_i(k+N-1|k)]'$. The FC-MPC optimization problem for subsystem i is given by Equation (4), in which each $\overline{\mathbf{u}}_j$, $\widehat{x}_j$, $j \in \mathbb{I}_M$ replaced by $\overline{\boldsymbol{\nu}}_j$, $\widehat{\omega}_j$, respectively.

For the augmented subsystem model (see Section 4.1), detectability of $(\widetilde{A}_i, \widetilde{C}_i)$ implies that a steady-state estimator gain $\widetilde{\mathcal{L}}_i$ exists such that $\widetilde{A}_i - \widetilde{A}_i \widetilde{\mathcal{L}}_i \widetilde{C}_i$ is stable. We have $\widetilde{e}_i^+ = (\widetilde{A}_i - \widetilde{A}_i \widetilde{\mathcal{L}}_i \widetilde{C}_i)\widetilde{e}_i$ and $\widetilde{Z}_i = \widetilde{\mathcal{L}}_i \widetilde{C}_i (\widetilde{A}_i - \widetilde{A}_i \widetilde{\mathcal{L}}_i \widetilde{C}_i)$, where $\widetilde{e}_i$ is the estimate error for the augmented subsystem model. Let $\mu^t_s = [x^t_{s_1}, \ldots, x^t_{s_M}]$. The evolution of the perturbed augmented system is given by

$$\begin{pmatrix} \widehat{x}_i \\ \widehat{d}_i \end{pmatrix}^+ = \widetilde{A}_i \begin{pmatrix} \widehat{x}_i \\ \widehat{d}_i \end{pmatrix} + \widetilde{B}_i u^p_i(\widehat{\mu} - \mu^t_s, 0) + \sum_{j \neq i}^{M} \widetilde{W}_{ij} u^p_j(\widehat{\mu} - \mu^t_s, 0) + \widetilde{Z}_i \widetilde{e}_i, \tag{14a}$$

$$\widetilde{e}_i^+ = (\widetilde{A}_i - \widetilde{A}_i \widetilde{\mathcal{L}}_i \widetilde{C}_i)\widetilde{e}_i, \quad z_i^{\text{sp},+} = z_i^{\text{sp}}, \qquad i \in \mathbb{I}_M, \tag{14b}$$

in which $\widehat{\mu} - \mu^t_s = [\widehat{x}_1 - x^t_{s_1}, \ldots, \widehat{x}_M - x^t_{s_M}]$. Define

$$\widetilde{\mathbb{D}}_C = \Big\{ \left((\widehat{x}_i, \widehat{d}_i, \widetilde{e}_i, z_i^{\text{sp}}),\ i \in \mathbb{I}_M\right) \mid \left((\widehat{x}_i^+, \widehat{d}_i^+, \widetilde{e}_i^+, z_i^{\text{sp},+}),\ i \in \mathbb{I}_M\right) \in \widetilde{\mathbb{D}}_C$$
$$\left((z_i^{\text{sp}}, \widehat{d}_i),\ i \in \mathbb{I}_M\right) \in \mathbb{D}_T, \quad \widehat{x}_i - x_{s_i}(z_i^{\text{sp}}, \widehat{d}_i) \in \mathbb{D}_{R_i},\ i \in \mathbb{I}_M \Big\} \qquad \text{domain of controller} \tag{15}$$

In Equation (15), $(\widehat{x}_i^+, \widehat{d}_i^+, \widetilde{e}_i^+, z_i^{\text{sp},+}), i \in \mathbb{I}_M$ is calculated using Equation (14). The set $\widetilde{\mathbb{D}}_C$ is positively invariant. The set $\widetilde{\mathbb{D}}_C$ represents the maximal positively invariant stabilizable set for distributed MPC (with target calculation, state estimation and regulation) under constant disturbances and time invariant setpoints.

Let $(x^t_{s_i}, u^t_{s_i})$ represent the state and input targets obtained for subsystem $i \in \mathbb{I}_M$ after $t \in \mathbb{I}_+, t \leq t_{\max}$ Algorithm 2 iterations. Let FC-MPC based on either Theorem 1 (stable systems) or Theorem 2 (unstable systems) be used for distributed regulation. Let Algorithm 1 be terminated after $p \in \mathbb{I}_+, p \leq p_{\max}$ iterations. The target shifted perturbed closed-loop system evolves according to

$$\widehat{\omega}_i(k+1) = A_i \widehat{\omega}_i(k) + B_i \nu_i(k) + \sum_{j \neq i}^{M} W_{ij} \nu_j(k) + \tau^x_i \widetilde{Z}_i \widetilde{e}_i(k), \tag{16a}$$

$$\widetilde{e}_i(k+1) = (\widetilde{A}_i - \widetilde{A}_i \widetilde{\mathcal{L}}_i \widetilde{C}_i)\widetilde{e}_i(k), \quad i \in \mathbb{I}_M, \tag{16b}$$

in which $\widehat{\omega}_i = \widehat{x}_i - x^t_{s_i}$, $\nu_i = u^p_i(\widehat{\mu} - \mu^t_s, 0)$ and $\tau^x_i [\widehat{x}_i{}', \widehat{d}_i{}']' = \widehat{x}_i$, $i \in \mathbb{I}_M$. The input injected into subsystem $i \in \mathbb{I}_M$, after p Algorithm 1 iterations and t Algorithm 2 iterations, is $u^p_i(\widehat{\mu} - \mu^t_s, 0) +$

$u^t_{s_i}$. The evolution of the disturbance estimate follows $\widehat{d}_i(k+1) = \widehat{d}_i(k) + \tau^d_i \widetilde{Z}_i \widetilde{e}_i(k),\ i \in \mathbb{I}_M$, where $\tau^d_i[\widehat{x}_i{}', \widehat{d}_i{}']' = \widehat{d}_i$.

Theorem 3. *Let $(\widetilde{A}_i, \widetilde{C}_i),\ i \in \mathbb{I}_M$ be detectable. The origin is an exponentially stable equilibrium for the target shifted perturbed closed-loop system given by Equation* (16)*, in which $(\widetilde{A}_i - \widetilde{A}_i\widetilde{\mathcal{L}}_i\widetilde{C}_i),\ i \in \mathbb{I}_M$ is stable and $\widehat{\omega}_i(0) = \widehat{x}_i(0) - x^t_{s_i}(0)$, for all $p = 1, 2, \ldots, p_{\max}$, $t = 1, 2, \ldots, t_{\max}$, $k \geq 0$ for all $\Big((\widehat{x}_i(0), \widehat{d}_i(0), \widetilde{e}_i(0), z^{\mathrm{sp}}_i),\ i \in \mathbb{I}_M\Big) \in \widetilde{\mathbb{D}}_C$.*

The proof is given in Appendix C.

The following lemma assures offset-free steady-state tracking performance.

Lemma 10. *Let Assumptions 1 to 3 hold. Let $(\widetilde{A}_i, \widetilde{C}_i),\ i \in \mathbb{I}_M$ be detectable, $(\widetilde{A}_i - \widetilde{A}_i\widetilde{\mathcal{L}}_i\widetilde{C}_i),\ i \in \mathbb{I}_M$ be stable and $n_{d_i} = n_{y_i},\ \forall\, i \in \mathbb{I}_M$. Also, let the input inequality constraints for each subsystem $i \in \mathbb{I}_M$ be inactive at steady state. If the closed-loop system under FC-MPC is stable, the FC-MPCs with subsystem-based estimators, local disturbance models and distributed target calculation, track their respective CV setpoints with zero offset at steady state i.e., $z^{\mathrm{sp}}_i = H_i y_i(\infty)$, where $y_i(\infty)$ is the output for subsystem i at steady state, and H_i satisfies Assumption 10.*

The proof is given in Appendix C.

6 Examples

6.1 Two reactor chain with nonadiabatic flash

A plant consisting of two continuous stirred tank reactors (CSTRs) and a nonadiabatic flash is considered. A schematic of the plant is shown in Figure 2. A description of the plant is available in [38, Section 8.2, p. 26]. A linear model for the plant is obtained by linearizing the plant around the steady state corresponding to the maximum yield of B. The constraints on the manipulated variables are given in Table 1. In decentralized and distributed MPC, there are 3 MPCs, one each for the two CSTRs and one for the nonadiabatic flash. Under centralized MPC, a single MPC controls the entire plant. A description of the MVs and CVs is provided in [38, Section 8.2, p. 26].

Table 1: Input constraints for Example 6.1. The symbol Δ represents a deviation from the corresponding steady-state value.

$-0.2 \leq \Delta F_0 \leq 0.2$	$-8 \leq \Delta Q_r \leq 8$
$-0.04 \leq \Delta F_1 \leq 0.04$	$-2 \leq \Delta Q_r \leq 2$
$-0.25 \leq \Delta D \leq 0.25$	$-8 \leq \Delta Q_b \leq 8$

The states and integrating disturbances for each subsystem are estimated from measurements. Input disturbance models are used to eliminate steady-state offset. The disturbance models employed in each MPC framework are given in Table 2. Under output feedback FC-MPC, the states and integrating disturbances for each subsystem are estimated from local measurements using subsystem-based Kalman filters. The steady-state targets are calculated in a distributed manner employing Algorithm 2. Two cases of distributed target calculation for FC-MPC are considered. In the first case, the distributed target calculation algorithm is terminated after 10 iterates, and in the second case, the distributed target calculation algorithm is iterated to convergence.

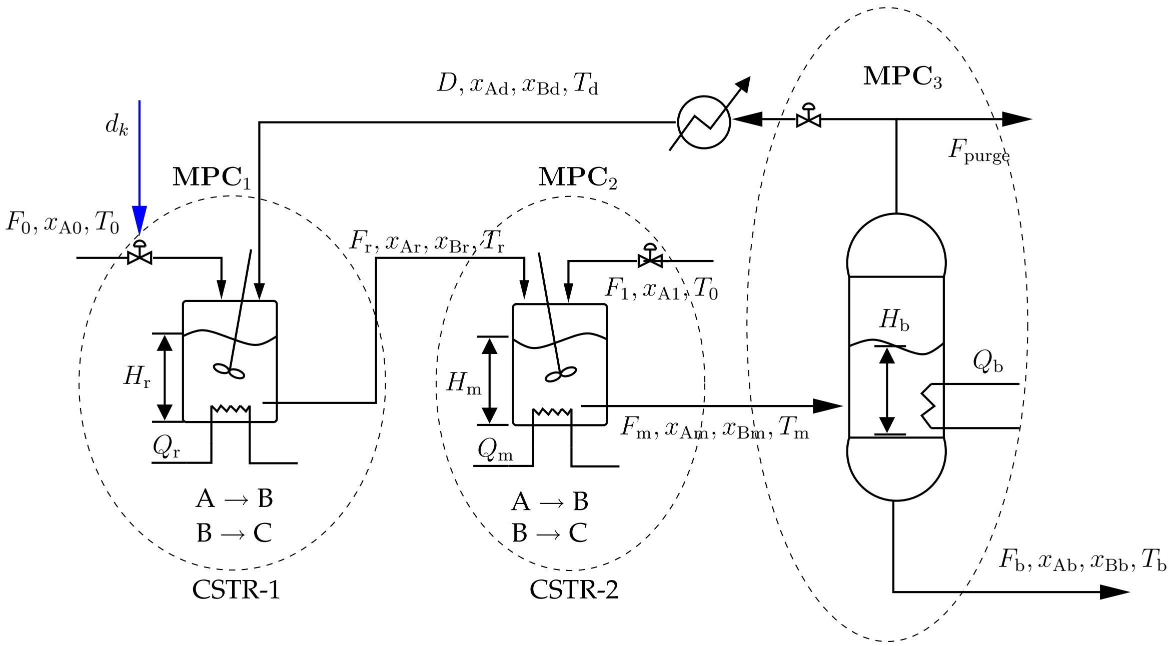

Figure 1: Two reactor chain followed by nonadiabatic flash. Vapor phase exiting the flash is predominantly A. Exit flows are a function of the level in the reactor/flash.

Table 2: Disturbance models (decentralized, distributed and centralized MPC frameworks) for Example 6.1.

$B_{11}^d = 0.5\begin{bmatrix} B_{11} & B_{11} \end{bmatrix}$ $B_1^d = \mathrm{vec}(B_{11}^d, 0, 0)$	$C_{11}^d = 0.5 I_{z_1}$ $C_1^d = 0.5 I_{z_1}$
$B_{22}^d = 0.5\begin{bmatrix} B_{22} & B_{22} \end{bmatrix}$ $B_2^d = \mathrm{vec}(0, B_{22}^d, 0)$	$C_{22}^d = 0.5 I_{z_2}$ $C_2^d = 0.5 I_{z_2}$
$B_{33}^d = 0.5\begin{bmatrix} B_{33} & B_{33} \end{bmatrix}$ $B_3^d = \mathrm{vec}(0, 0, B_{33}^d)$ $B_d = \mathrm{diag}(B_1^d, B_2^d, B_3^d)$	$C_{33}^d = 0.5 I_{z_3}$ $C_3^d = 0.5 I_{z_3}$ $C_d = \mathrm{diag}(C_1^d, C_2^d, C_3^d)$

A feed flowrate disturbance affects CSTR-1 from time = 30. As a result of this flowrate disturbance, the feed flowrate to CSTR-1 is increased by 5% (relative to the steady-state value). The disturbance rejection performance of centralized MPC, decentralized MPC and FC-MPC is investigated for the described disturbance scenario. A control horizon $N = 15$ is used for each MPC. The sampling interval is 1.5. The weight for each CV is 10 and the weight for each MV is 1. The performance of the different MPC frameworks rejecting the feed flowrate disturbance to CSTR-1 (d_k in Figure 1) is shown in Figure 2. The resulting temperature and cooling duty changes are small and therefore, not shown. The closed-loop control costs incurred by the different MPC frameworks are compared in Table 3.

Under decentralized MPC, the feed flowrate disturbance causes closed-loop instability. With the centralized MPC and FC-MPC frameworks, the system is able to reject the feed flowrate disturbance. The feed flowrate disturbance d_k to CSTR-1 causes an increase in H_{r}. In the FC-MPC framework, MPC-1 lowers F_0 to compensate for the extra material flow into CSTR-1. MPC-3 cooperates with MPC-1 and helps drive H_r back to its setpoint by decreasing the recycle flowrate D to CSTR-1. The initial increase in H_{r} results in an increase in F_{r}, which in turn increases H_{m}. Subsequently, F_{m} and hence, H_{b} also increase. To compensate for the initial increase in H_{m}, MPC-2 decreases F_1. The initial increase in H_{b} is due to an increase in F_{m} (MPC-2) and decrease in D (MPC-3). To lower H_{b}, MPC-3 subsequently increases D. MPC-1 continues to steer H_{r} to its setpoint, inspite of an increase in D (by MPC-3), through a corresponding (further) reduction in F_0.

The performance loss incurred under FC-MPC, with Algorithm 1 terminated after 1 cooperation-based iterate and Algorithm 2 terminated after 10 iterates, is $\sim 72\%$ relative to centralized MPC. If the distributed target calculation algorithm (Algorithm 2) is iterated to convergence, the performance loss relative to centralized MPC reduces to $\sim 38\%$. If both Algorithm 1 and Algorithm 2 are terminated after 10 iterates, the performance loss relative to centralized MPC is $\sim 43\%$. Iterating Algorithm 2 to convergence, and terminating Algorithm 1 after 10 iterates, improves performance to within 1% of centralized MPC performance.

Table 3: Closed-loop performance comparison of centralized MPC, decentralized MPC and FC-MPC. The distributed target calculation algorithm (Algorithm 2) is used to determine steady-state subsystem input, state and output target vectors in the FC-MPC framework.

	$\Lambda_{\mathrm{cost}} \times 10^2$	$\Delta\Lambda_{\mathrm{cost}}\%$
Cent-MPC	4.14	$--$
Decent-MPC	∞	$--$
FC-MPC (1 iterate, targ=conv)	5.74	38.4%
FC-MPC (1 iterate, targ=10)	7.12	71.2%
FC-MPC (10 iterates, targ=conv)	4.17	0.62%
FC-MPC (10 iterates, targ=10)	5.93	43.2%

6.2 Irrigation Canal Network

The key to agricultural productivity and the goal of irrigation canal networks is to provide the right quantity of water at the right time and place. The need for flexible, "on-demand" schedules has motivated the need for automatic control of these water networks. Each irrigation canal

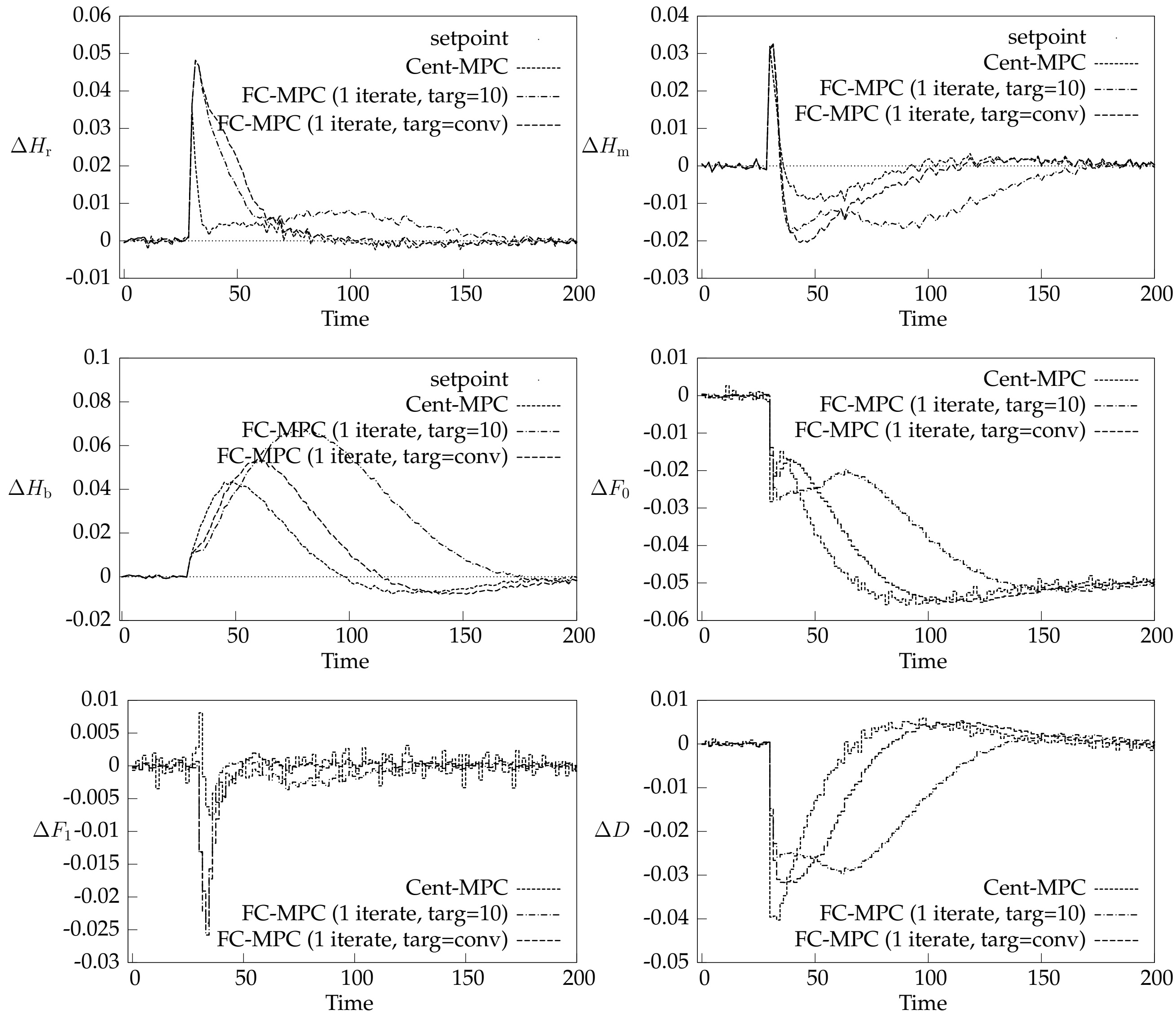

Figure 2: Disturbance rejection performance of centralized MPC, decentralized MPC and FC-MPC. For the FC-MPC framework, 'targ=conv' indicates that the distributed target calculation algorithm is iterated to convergence. The notation 'targ=10' indicates that the distributed target calculation algorithm is terminated after 10 iterates.

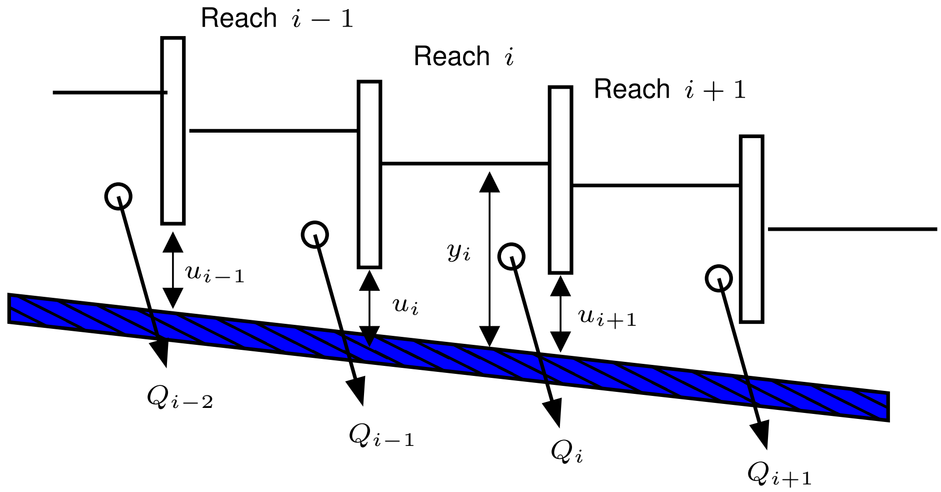

Figure 3: Structure of an irrigation canal. Each canal consists of a number of interconnected reaches.

consists of a fixed number of reaches that are interconnected through control gates. In reach i (see Figure 3), the downstream water level y_i is controlled by manipulating the gate opening u_i. However the water level y_i is also affected by the gate opening u_{i+1} in reach $i+1$. At the downstream end of each reach i, an off-take discharge Q_i supplies water to meet local demands. For each reach, the off-take discharge is dictated by the local water demand. Variations in the off-take discharge are disturbances for the system. Representative publications on the modeling of canal networks include [11; 30]. Typically, different sections of the irrigation canal network are administered by different governing bodies (e.g., different municipalities) making centralized control impractical and unrealizable. A decentralized control formulation in which each canal reach employs a local controller to regulate water levels may realize poor closed-loop performance as a result of the interaction between adjacent reaches.

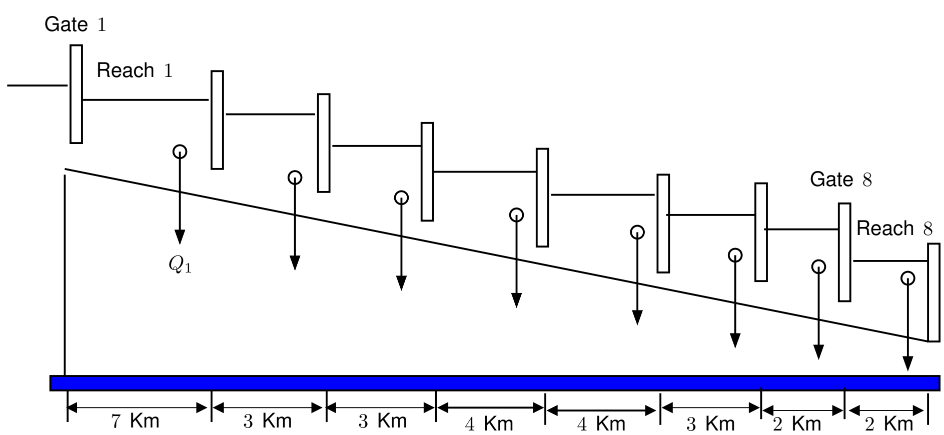

Figure 4: Profile of ASCE test canal 2 [6]. Total canal length 28 km.

The example we consider here is canal 2 of the test cases established by the ASCE task com-

Table 4: Gate opening constraints for Example 6.2. The symbol Δ denotes a deviation from the corresponding steady-state value.

$-1.5 \leq \Delta u_1 \leq 1.5$
$-1.5 \leq \Delta u_2 \leq 1.5$
$-0.4 \leq \Delta u_3 \leq 0.4$
$-0.1 \leq \Delta u_4 \leq 0.1$
$-0.25 \leq \Delta u_5 \leq 0.25$
$-0.15 \leq \Delta u_6 \leq 0.15$
$-0.1 \leq \Delta u_7 \leq 0.1$
$-0.1 \leq \Delta u_8 \leq 0.1$

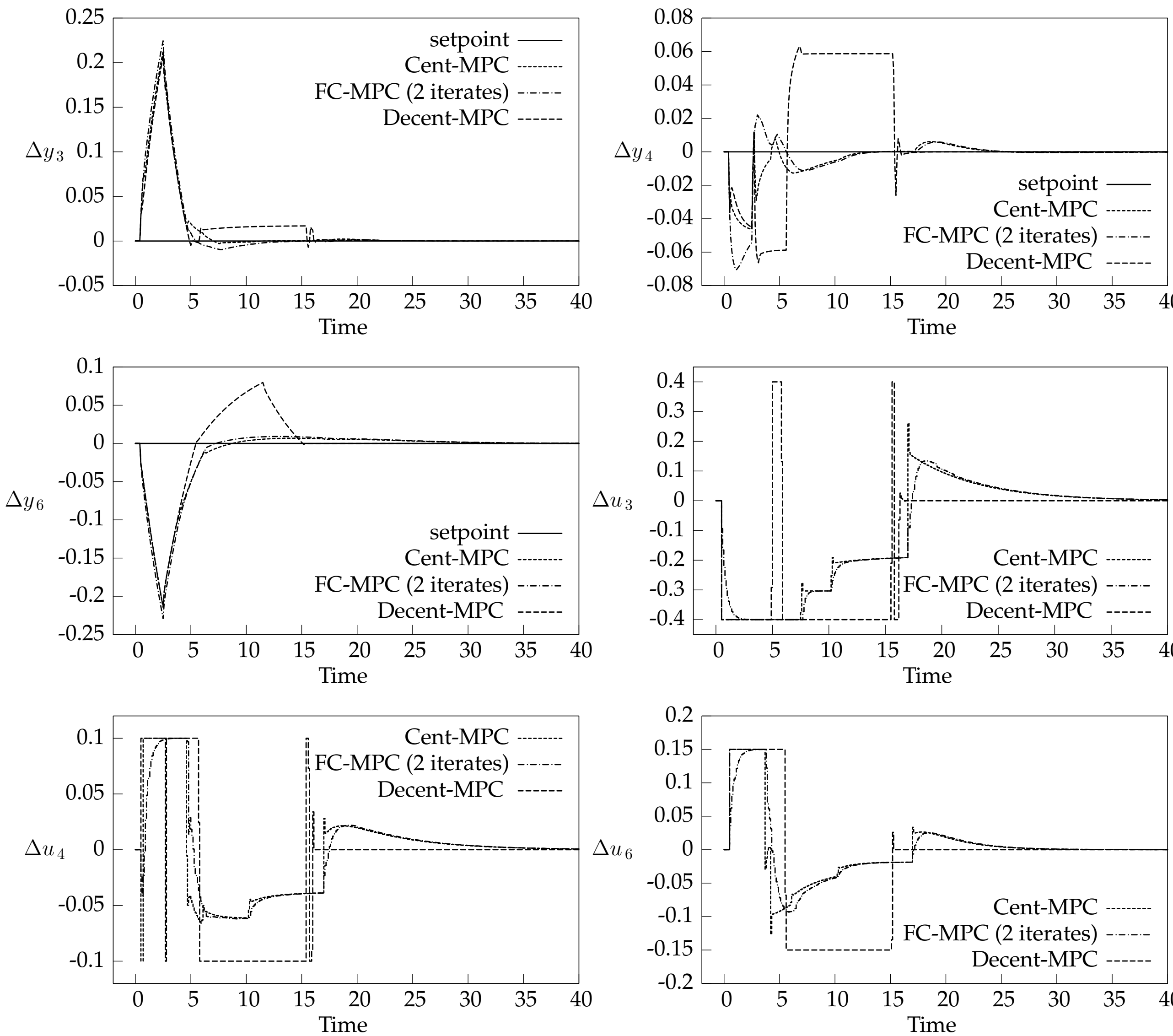

Figure 5: Control of ASCE test canal 2. Water level control for reaches $3, 4$ and 6.

mittee [6]. The canal under consideration is fed by a constant water level reservoir at its head. The canal consists of 8 interconnected reaches with the downstream end closed (Figure 4). Between times 0.5 hrs and 2.5 hrs, an off-take discharge disturbance affects reaches $1-8$. During this time, reaches $1,3,5$ and 7 experience an off-take discharge disturbance of $2.5\ \mathrm{m}^3/\mathrm{min}$ and simultaneously, a discharge disturbance $-2.5\ \mathrm{m}^3/\mathrm{min}$ affects reaches $2,4,6$ and 8. The closed-loop performance of centralized MPC, decentralized MPC and FC-MPC, rejecting this off-take discharge disturbance is assessed. The permissible gate opening Δu_i (deviation w.r.t to steady state) for each reach $i, i \in \{1,2,\ldots,8\}$ is given in Table 4. In the decentralized and distributed MPC frameworks, there are 8 MPCs, one for each reach.

Table 5: Closed-loop performance of centralized MPC, decentralized MPC and FC-MPC rejecting the off-take discharge disturbance in reaches $1-8$. The distributed target calculation algorithm (Algorithm 2) is iterated to convergence.

	$\Lambda_{\mathrm{cost}} \times 10^4$	$\Delta\Lambda_{\mathrm{cost}}\%$
Cent-MPC	4.58	$--$
Decent-MPC	7.81	70.5%
FC-MPC (1 iterate)	7.41	61.8
FC-MPC (2 iterates)	5.88	28.4%
FC-MPC (5 iterates)	4.99	9.0%
FC-MPC (10 iterates)	4.72	3.1%

In each MPC framework, the state and constant disturbances are estimated using a steady-state Kalman filter. Output disturbance models are used to eliminate steady-state offset due to the unmeasured off-take discharge disturbances. For FC-MPC, the distributed target calculation algorithm is iterated to convergence. The performance of the different MPCs, rejecting the off-take discharge disturbance in reaches $3,4$ and 6 is shown in Figure 5. For each MPC, $Q_i = 10, R_i = 0.1, i \in \{1,2,\ldots,8\}$. The sampling rate is 0.1 hrs (or every 6 minutes) and the control horizon N for each MPC is 30.

Based on control costs calculated at steady state (Table 5), decentralized MPC leads to a control performance loss of $\sim 70\%$ relative to centralized MPC performance. In the FC-MPC framework, with Algorithm 1 terminated after a single iterate, the incurred performance loss w.r.t centralized MPC is $\sim 62\%$. If the FC-MPC algorithm is terminated after 2 cooperation-based iterates, the performance loss reduces to $\sim 29\%$ of centralized MPC performance. The FC-MPC framework achieves performance that is within 3.25% of centralized MPC performance if 10 cooperation-based iterates are allowed.

7 Discussion and conclusions

The primary contribution of this work is an output feedback distributed MPC framework with guaranteed feasibility, optimality and perturbed closed-loop stability properties. A distributed state estimation strategy was proposed for estimating the subsystem states using local measurements. An attractive feature of the proposed distributed estimator design procedure (Section 2.1) is that it requires only local process data. The distributed estimation strategies presented here do not require a 'master' processor. The trade-off is the suboptimality of the generated estimates; the obtained estimates, however, converge to the true subsystem states exponentially for the nominal model. The FC-MPC algorithm (Algorithm 1) presented in [38] is used for distributed regulation. Closed-loop stability under decaying perturbations for all (Algorithm 1) iteration numbers is established. The perturbed closed-loop stability result guarantees that the distributed estimator-distributed regulator assembly is stabilizing under intermediate termination of the FC-MPC algorithm.

A disturbance modeling framework that uses local integrating disturbances was employed. This choice of local integrating disturbances is motivated by considerations of simplicity and practical convenience. A simple rank test is necessary and sufficient to verify suitability of postulated subsystem disturbance models. Next, a distributed target calculation algorithm that computes steady-state input, state and output targets at the subsystem level was described. All iterates generated by the distributed target calculation algorithm are feasible steady states. Also, the target cost function is monotonically nonincreasing with iteration number. The attributes described above allow intermediate termination of the distributed target calculation algorithm. A maximal positively invariant stabilizable set for distributed MPC, with state estimation, target calculation and regulation, was defined. This positively invariant set characterizes system state, disturbance, estimate error and setpoint quadruples for which the system can be stabilized using the distributed MPC control law. Zero-offset control at steady state is established for the set of subsystem-based MPCs under the assumption that the input constraints for each subsystem are inactive at steady state. An interesting result, which follows from Lemmas 6 and 10, is that disturbance models that achieve zero-offset steady-state control under decentralized MPC are sufficient to achieve zero-offset steady-state control in the FC-MPC framework.

Two examples were presented to illustrate the effectiveness of the proposed output feedback distributed MPC framework. In the first example, control of a chemical plant was investigated in the presence of a feed flowrate disturbance. Decentralized MPC is unable to reject the disturbance and results in closed-loop instability. In this example, the distributed target calculation algorithm (for FC-MPC) was terminated at an intermediate iterate. FC-MPC is able to reject the disturbance and achieves zero offset steady-state control performance for all values of distributed target calculation and distributed regulation algorithm iteration numbers. Next, FC-MPC was employed to reject a discharge disturbance in an irrigation canal. Local output disturbance models were used to achieve zero offset steady-state control performance. The first iterate was observed to improve performance marginally ($\sim 9\%$) compared to decentralized MPC. A second iterate, however, leads to a significant improvement in performance ($\sim 41\%$) compared to decentralized MPC. For this example, it is recommended that at least two iterates per sampling interval be performed.

Implementation. The structure of the FC-MPC framework with distributed estimation, local disturbance modeling and distributed target calculation is shown in Figure 6. Each subsystem uses a local Kalman filter to estimate its states and integrating disturbances. The only external information required are the inputs injected into the interconnected subsystems. This input information, however, is available in the regulator and consequently, no information transfer between subsystems is needed at the estimation level. Next, the targets are calculated locally. The input target is relayed to all interacting subsystems after each iterate (Algorithm 2). The setpoint for subsystem CVs, local integrating disturbances and the decentralized state target need not be communicated to interconnected subsystems. For distributed regulation, the subsystem state estimate is commu-

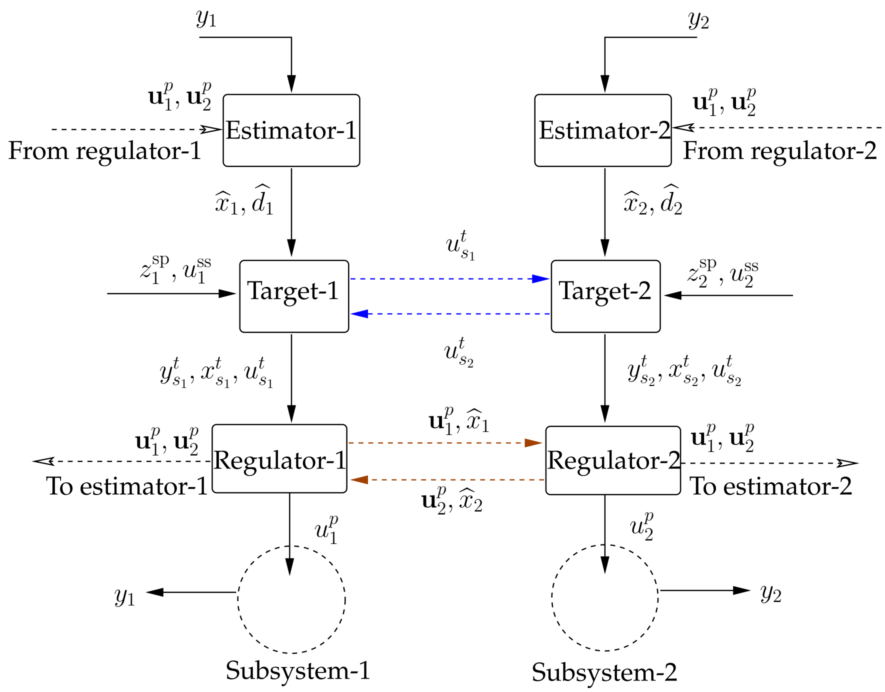

Figure 6: Structure of output feedback FC-MPC.

nicated to all interconnected subsystems at each k; the recalculated input trajectories are broadcast to interacting subsystems after each iterate. For each subsystem $i \in \mathbb{I}_M$, by setting $w_i = 1$ and $w_j = 0,\ \forall\, j \neq i$ in the FC-MPC regulator optimization problem and $A_{ij}, C_{ij} = 0$, $\forall\, j \neq i$ in the estimator and the target optimization problem, and by switching off the communication between the subsystems, we revert to decentralized MPC.

8 Acknowledgment

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Appendices

A State estimation for FC-MPC

Proof for Lemma 1. Let T be a similarity transform for the LTI system (A_m, B_m, C_m, G_m) with (A_m, B_m) stabilizable and (A_m, C_m) detectable. Let the transformed LTI system be

$$(\widetilde{A}_m, \widetilde{B}_m, \widetilde{C}_m, \widetilde{G}_m) = (TA_mT^{-1}, TB_m, C_mT^{-1}, TG_m).$$

We know from the Hautus lemma [33] that

$$\text{rank}(\mathbb{H}(\lambda)) = \text{rank}\begin{bmatrix}\lambda I - A_m \\ C_m\end{bmatrix} = n,\ \forall\,|\lambda| \geq 1$$

From the definition of T and $\mathbb{H}(\lambda)$, we have

$$\mathbb{H}(\lambda) = \begin{bmatrix}\lambda I - A_m \\ C_m\end{bmatrix} = \begin{bmatrix}\lambda I - T^{-1}\widetilde{A}_mT \\ \widetilde{C}_mT\end{bmatrix} = \begin{bmatrix}T^{-1} & \\ & I\end{bmatrix}\begin{bmatrix}\lambda I - \widetilde{A}_m \\ \widetilde{C}_m\end{bmatrix}T$$

Let $\widetilde{\mathbb{H}}[\lambda] = \begin{bmatrix}\lambda I - \widetilde{A}_m \\ \widetilde{C}_m\end{bmatrix}$. Therefore, $\widetilde{\mathbb{H}}[\lambda] = \begin{bmatrix}T & \\ & I\end{bmatrix}\mathbb{H}(\lambda)\,T^{-1}$. Suppose $(\widetilde{A}_m, \widetilde{C}_m)$ is not detectable. By assumption, there exists $\lambda_1,\ |\lambda_1| \geq 1$ and z such that $\widetilde{\mathbb{H}}[\lambda_1]z = 0, z \neq 0$, which gives

$$\begin{bmatrix}T & \\ & I\end{bmatrix}\mathbb{H}[\lambda_1]\,T^{-1}z = 0,\ z \neq 0$$

Let $v = T^{-1}z$. Since $z \neq 0$ and T is full rank, $v \neq 0$. This gives $\mathbb{H}[\lambda_1]v = 0, v \neq 0$, which contradicts detectability of (A_m, C_m). The arguments establishing the implication $(\widetilde{A}_m, \widetilde{C}_m)$ detectable $\implies$ (A_m, C_m) detectable are similar to those used earlier with T replaced by T^{-1}. Since stabilizability of $(\widetilde{A}_m, \widetilde{B}_m) \equiv$ detectability of $(\widetilde{A}_m{}', \widetilde{B}_m{}')$, stabilizability is also invariant under a similarity transformation. □

Theorem 4. *Let*

$$\mathbb{A} = \begin{pmatrix} \mathcal{A} & \\ & \mathcal{A}_s \end{pmatrix} \in \mathbb{R}^{(n+n_s)\times(n+n_s)}, \quad \mathbb{C} = \begin{pmatrix} \mathcal{C} & \mathcal{C}_s \end{pmatrix} \in \mathbb{R}^{n_y \times (n+n_s)}, \tag{17}$$

in which $\mathcal{A}_s$ is stable, $\mathcal{A} \in \mathbb{R}^{n\times n}$ and $\mathcal{C} \in \mathbb{R}^{n_y \times n}$. The pair $(\mathcal{A}, \mathcal{C})$ is detectable if and only if $(\mathbb{A}, \mathbb{C})$ is detectable.

Proof. From the Hautus lemma for detectability [33, p. 318], $(\mathcal{A}, \mathcal{C})$ is detectable iff $\operatorname{rank}\left(\begin{bmatrix} \lambda I - \mathcal{A} \\ \mathcal{C} \end{bmatrix}\right) = n,\ \forall\, |\lambda| \geq 1$.

$(\mathcal{A}, \mathcal{C})$ **detectable** $\implies$ $(\mathbb{A}, \mathbb{C})$ **detectable.** Consider $|\lambda| \geq 1$. Detectability of $(\mathcal{A}, \mathcal{C})$ implies the columns of $\begin{bmatrix} \lambda I - \mathcal{A} \\ \mathcal{C} \end{bmatrix}$ are independent. Hence, $\begin{bmatrix} \lambda I - \mathcal{A} \\ 0 \\ \mathcal{C} \end{bmatrix}$ has independent columns. Since $\mathcal{A}_s$ is stable, the columns of $\lambda I - \mathcal{A}_s$ are independent, which implies the columns of $\begin{bmatrix} 0 \\ \lambda I - \mathcal{A}_s \\ \mathcal{C}_s \end{bmatrix}$ are also independent. Due to the positions of the zeros, the columns of

$$\begin{bmatrix} \lambda I - \mathcal{A} & 0 \\ 0 & \lambda I - \mathcal{A}_s \\ \mathcal{C} & \mathcal{C}_s \end{bmatrix}$$

are also independent. Hence, $(\mathbb{A}, \mathbb{C})$ is detectable.

$(\mathbb{A}, \mathbb{C})$ **detectable** $\implies$ $(\mathcal{A}, \mathcal{C})$ **detectable.** We have from the Hautus lemma for detectability that the columns of

$$\begin{bmatrix} \lambda I - \mathcal{A} & \\ & \lambda I - \mathcal{A}_s \\ \mathcal{C} & \mathcal{C}_s \end{bmatrix}$$

are independent for all $|\lambda| \geq 1$. The columns of $\begin{bmatrix} \lambda I - \mathcal{A} \\ 0 \\ \mathcal{C} \end{bmatrix}$ are, therefore, independent. Hence, the columns of $\begin{bmatrix} \lambda I - \mathcal{A} \\ \mathcal{C} \end{bmatrix}$ are independent, which gives $(\mathcal{A}, \mathcal{C})$ is detectable. □

Proof for Lemma 2. Let $\mathcal{A} = A_{ii}$, $\mathcal{C} = C_{ii}$, $\mathcal{A}_s = \operatorname{diag}(A_{i1}, \ldots, A_{i(i-1)}, A_{i(i+1)}, \ldots, A_{iM})$ and $\mathcal{C}_s = [C_{i1}, \ldots, C_{i(i-1)}, C_{i(i+1)}, \ldots, C_{iM}]$. Also, let $\mathbb{A}$, $\mathbb{C}$ be given by Equation (17). We note that $A_i = U\mathbb{A}U$, $C_i = \mathbb{C}U$, in which U is a unitary matrix (hence a similarity transform). Invoking Theorem 4 and Lemma 1, we have the required result. □

Proof for Lemma 3. $(\widehat{A}_i, \widehat{G}_i)$ **stabilizable** $\implies$ (A_i^o, G_i^o) **stabilizable.** By assumption, we have

$$\operatorname{rank}\begin{bmatrix} \lambda I - A_i^o & 0 & G_i^o \\ -A_i^{12} & \lambda I - A_i^{\overline{o}} & 0 \end{bmatrix} = n_i = n_i^o + n_i^{\overline{o}},\ \forall\, |\lambda| \geq 1,$$

in which $A_i^{\overline{o}} \in \mathbb{R}^{n_i^{\overline{o}} \times n_i^{\overline{o}}}$. Consider $|\lambda| \geq 1$. From the rank condition above, we have that the rows of $[\lambda I - A_i^o, 0, G_i^o]$ are independent. Hence, the rows of $\begin{bmatrix}\lambda I - A_i^o & G_i^o\end{bmatrix}$ are independent *i.e.*, (A_i^o, G_i^o) is stabilizable.

(A_i^o, G_i^o) **stabilizable** $\implies$ $(\widehat{A}_i, \widehat{G}_i)$ **stabilizable.** Since (A_i^o, G_i^o) is stabilizable, the rows of $\begin{bmatrix}\lambda I - A_i^o & G_i^o\end{bmatrix}$ are independent for all $|\lambda| \geq 1$. Hence, the rows of $\begin{bmatrix}\lambda I - A_i^o & 0 & G_i^o\end{bmatrix}$ are also independent. From Lemma 2, (A_i, C_i) is detectable. Since (A_i, C_i) is detectable, its observability canonical form $(\widehat{A}_i, \widehat{C}_i)$ is also detectable (Lemma 1). From Equation (2), $A_i^{\overline{o}}$ is stable. The rows of $\lambda I - A_i^{\overline{o}}$ are independent, which implies the rows of $\begin{bmatrix}-A_i^{12} & \lambda I - A_i^{\overline{o}} & 0\end{bmatrix}$ are also independent. Due to the positions of the zeros, the rows of

$$\begin{bmatrix}\lambda I - A_i^o & 0 & G_i^o \\ -A_i^{12} & \lambda I - A_i^{\overline{o}} & 0\end{bmatrix}$$

are independent, which gives $(\widehat{A}_i, \widehat{G}_i)$ is stabilizable. □

B Closed-loop properties of FC-MPC under output feedback

Lemma 11 ([5]). *Suppose Z is a positive semidefinite $n \times n$ matrix and a, b are n-dimensional vectors. Then given $\delta > 0$,*

$$(a+b)'Z\,(a+b) \leq (1+\delta)\,a'Za + \left(1 + \frac{1}{\delta}\right) b'Zb \tag{18}$$

Lemma 12. *Let $\mathcal{A}x = b$ be a system of linear equations with $\mathcal{A} \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, $m \leq n$. Consider $\mathcal{X} \subset \mathbb{R}^n$ nonempty, compact, convex with $0 \in \operatorname{int}(\mathcal{X})$. The set $\mathcal{B} \subseteq \operatorname{range}(\mathcal{A})$ is defined as $\mathcal{B} = \{b \mid \mathcal{A}x = b, x \in \mathcal{X}\}$. For every $b \in \mathcal{B}$, $\exists\ \overline{x}(b)$ dependent on b, and $K > 0$ independent of b such that $\mathcal{A}\overline{x}(b) = b$, $\overline{x}(b) \in \mathcal{X}$ and $\|\overline{x}(b)\| \leq K\|b\|$.*

A proof is given in [38, Appendix A].

Definition 1 (Hölder's inequality). For any set of nonnegative quantities a_i and b_i, $i = 1, 2, \ldots, n$, we have

$$(a_1^p + a_2^p + \ldots + a_n^p)^{1/p}(b_1^q + b_2^q + \ldots + b_n^q)^{1/q} \geq a_1b_1 + a_2b_2 + \ldots + a_nb_n$$

in which p and q are related by

$$\frac{1}{p} + \frac{1}{q} = 1$$

Corollary 1.1. For any set of nonnegative quantities a_i, $i = 1, 2, \ldots, n$, $n^{p-1}\left(\sum_{i=1}^n a_i^p\right) \geq \left(\sum_{i=1}^n a_i\right)^p$.

Proof. The result follows by choosing $b_i = 1$, $i = 1, 2, \ldots, n$ in Hölder's inequality (Definition 1), and noting that $\frac{p}{q} = p - 1$. □

B.1 Preliminaries

Nominal closed-loop subsystem. Let Algorithm 1 be terminated after $p \in \mathbb{I}_+$ iterates. The evolution of each nominal closed-loop subsystem $i \in \mathbb{I}_M$ follows $x_i^+ = A_ix_i + B_iu_i^p(\mu, 0) + \sum_{j \neq i} W_{ij}u_j^p(\mu, 0) = F_i^p(\mu)$, in which $u_i^p(\mu, 0)$ is the control law for subsystem i.

Perturbed closed-loop subsystem. Let $e_i = x_i - \widehat{x}_{i,\ominus}$ denote the current estimate error for subsystem $i \in \mathbb{I}_M$. The symbol $\widehat{x}_{i,\ominus}$ denotes the estimate of x_i before current measurement y_i is available; $\widehat{x}_i$ represents the estimate of x_i after y_i is available. Let e_i^+ denotes the estimate error at the subsequent time step.

Assumption 12. $e_i^+ = \mathcal{A}_i^L e_i,\ |\lambda_{\max}(\mathcal{A}_i^L)| < 1, i \in \mathbb{I}_M$,

For Algorithm 1 terminated after p iterates, the control law for subsystem $i \in \mathbb{I}_M$ is $u_i^p(\widehat{\mu}, 0)$ (see Section 3.1). We have the following equations for the filter for subsystem i

$$\widehat{x}_i = \widehat{x}_{i,\ominus} + \mathfrak{L}_i(y_i - C_i\widehat{x}_{i,\ominus}), \quad \widehat{x}_{i,\ominus}^+ = A_i\widehat{x}_i + B_i u_i^p(\widehat{\mu}, 0) + \sum_{j \neq i}^{M} W_{ij} u_j^p(\widehat{\mu}, 0),$$

in which $\widehat{x}_{i,\ominus}^+$ represents an estimate of the successor subsystem state x_i^+ before new measurement y_i^+ is available and $\mathfrak{L}_i, i \in \mathbb{I}_M$ is the filter gain. For each subsystem $i \in \mathbb{I}_M$, we have

$$\xi_i(1) = \widehat{x}_i^+ - \widehat{x}_{i,\ominus}^+ = (\widehat{x}_{i,\ominus}^+ + \mathfrak{L}_i C_i e_i^+) - \widehat{x}_{i,\ominus}^+ = Z_i e_i, \quad Z_i = \mathfrak{L}_i C_i \mathcal{A}_i^L, \tag{19}$$

in which $\widehat{x}_i^+$ represents an estimate of x_i^+ after y_i^+ is available. Consider Figure 7. Let $\mathbf{x}_i^p = [\rho_i^p(1)', \rho_i^p(2)', \ldots]'$ be the state trajectory for subsystem $i \in \mathbb{I}_M$ generated by $\mathbf{u}_1^p, \ldots, \mathbf{u}_M^p$ and initial subsystem state $\widehat{x}_i$ (trajectory $\mathfrak{A}_i^p$ in Figure 7). We have $\rho_i^p(1) = \widehat{x}_{i,\ominus}^+, i \in \mathbb{I}_M$. The evolution of $\rho_i^p(j), j \geq 1$ in $\mathfrak{A}_i^p$ is

$$\rho_i^p(j) = A_i^{j-1}\rho_i^p(1) + \sum_{l=1}^{j-1} A_i^{j-1-l} B_i u_i^p(\widehat{\mu}, l) + \sum_{s \neq i}^{M} \sum_{l=1}^{j-1} A_i^{j-1-l} W_{is} u_s^p(\widehat{\mu}, l) \tag{20}$$

The state estimate for subsystem $i \in \mathbb{I}_M$ at the subsequent time step is $\widehat{x}_i^+ = \rho_i^p(1) + Z_i e_i$. Let $z_i(1) = \widehat{x}_i^+$. For each $i \in \mathbb{I}_M$, let $\mathbf{w}_i = [w_i(1)', w_i(2)', \ldots]'$, $w_i(j) \in \Omega_i,\ j \geq 1$ be an admissible input trajectory from $z_i(1)$. Let $\mathbf{z}_i = [z_i(2)', z_i(3), \ldots]'$ be the state trajectory for subsystem $i \in \mathbb{I}_M$ generated by $\mathbf{w}_1, \ldots, \mathbf{w}_M$ and initial subsystem state $z_i(1)$ (trajectory $\mathfrak{B}_i^0$ in Figure 7). For $z_i(j)$ in $\mathfrak{B}_i^0$, we write

$$z_i(j) = A_i^{j-1} z_i(1) + \sum_{l=1}^{j-1} A_i^{j-1-l} B_i w_i(l) + \sum_{s \neq i}^{M} \sum_{l=1}^{j-1} A_i^{j-1-l} W_{is} w_s(l) \tag{21}$$

Define $\xi_i(j) = z_i(j) - \rho_i^p(j), v_i(j) = w_i(j) - u_i^p(\widehat{\mu}, j),\ j \geq 1$ and all $i \in \mathbb{I}_M$. For $j = 1$, we know from Equation (19) that $\xi_i(1) = Z_i e_i,\ i \in \mathbb{I}_M$. For $j > 1$, we have from Equations (20) and (21) that

$$\xi_i(j) = A_i^{j-1}\xi_i(1) + \sum_{l=1}^{j-1} A_i^{j-1-l} B_i v_i(l) + \sum_{s \neq i}^{M} \sum_{l=1}^{j-1} A_i^{j-1-l} W_{is} v_s(l) \tag{22}$$

For Algorithm 1 terminated after p iterates, the evolution of each perturbed closed-loop subsystem $i \in \mathbb{I}_M$ follows

$$\widehat{x}_i^+ = A_i\widehat{x}_i + B_i u_i^p(\widehat{\mu}, 0) + \sum_{j \neq i}^{M} W_{ij} u_j^p(\widehat{\mu}, 0) + Z_i e_i, \quad e_i^+ = \mathcal{A}_i^L e_i \tag{23}$$

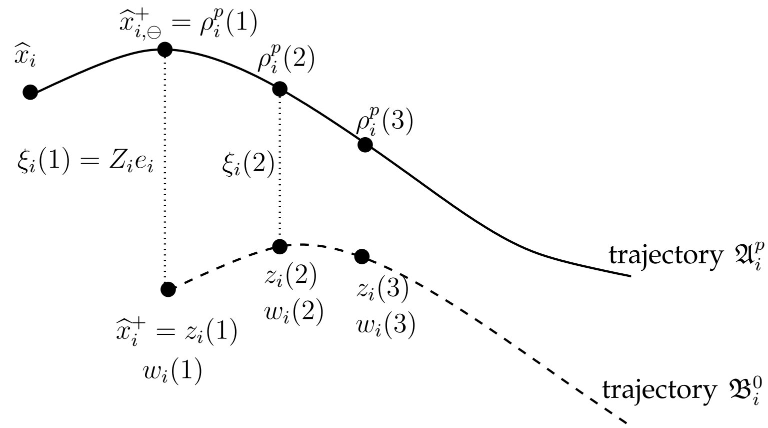

Figure 7: Trajectory $\mathfrak{A}^p_i$ is the state trajectory for subsystem i generated by $\mathbf{u}^p_1, \ldots, \mathbf{u}^p_M$ and initial subsystem state $\widehat{x}_i$. The state trajectory $\mathfrak{B}^0_i$ for subsystem i is generated by $\mathbf{w}_1, \ldots, \mathbf{w}_M$ from initial state $z_i(1)$.

B.2 Main result

Let $\widehat{\mu}^+ = [\widehat{x}^+_1, \ldots \widehat{x}^+_M]$ and $p, q \in \mathbb{I}_+$. For the set of estimated subsystem states $\widehat{\mu}$, we assume (WLOG) that Algorithm 1 is terminated after p iterates. At the subsequent time step with estimated state $\widehat{\mu}^+$, let q (possibly different from p) iterates be performed. Let the distributed MPC control law $u^p_i(\widehat{\mu}, 0),\ i \in \mathbb{I}_M$ be defined for $\widehat{\mu} \in \mathbb{X}_0$. Define $\mathbb{X}_u = \{\widehat{\mu} \,|\, \widehat{\mu} \in \mathbb{X}_0, u^p_i(\widehat{\mu}, 0) \in \Omega_i,\ i \in \mathbb{I}_M\}$.

Assumption 13. For the nominal closed-loop system $x^+_i = F^p_i(\mu), i \in \mathbb{I}_M$, $J^p_N(\mu) = \Phi([\mathbf{u}^p_1, \ldots, \mathbf{u}^p_M]\ ; \mu)$ is a Lyapunov function satisfying

$$a_r \sum_{i=1}^{M} \|x_i\|^2 \leq J^p_N(\mu) \leq b_r \sum_{i=1}^{M} \|x_i\|^2 \tag{24a}$$

$$\Delta J_N(\mu) \leq -c_r \sum_{i=1}^{M} \|x_i\|^2 \tag{24b}$$

in which $a_r, b_r, c_r > 0$ and $\Delta J_N(\mu) = J^q_N(\mu^+) - J^p_N(\mu)$.

Define the set

$$\mathbb{Z} = \{\, ((\widehat{x}_i, e_i),\ i \in \mathbb{I}_M) \mid \big((\widehat{x}^+_i, e^+_i),\ i \in \mathbb{I}_M\big) \in \mathbb{Z},\ \widehat{\mu} \in \mathbb{X}_u\}, \tag{25}$$

in which $(\widehat{x}^+_i, e^+_i)$ is given by Equation (23). Let $((\widehat{x}_i(0), e_i(0)),\ i \in \mathbb{I}_M)$ represent the set of initial (estimated) subsystem states and initial estimate errors, respectively.

Theorem 5. *Let Assumptions 6, 12 and 13 hold. Consider the auxiliary system*

$$\xi_i(j+1) = A_i \xi_i(j) + B_i v_i(j) + \sum_{l \neq i}^{M} W_{il} v_l(j), \quad v_i(j) + u^p_i(\widehat{\mu}(0), j) \in \Omega_i,\ \forall\, i \in \mathbb{I}_M,\ j \geq 1,$$

with initial condition $\xi_i(1) = Z_i e_i(0)$*. Suppose a set of perturbation trajectories* $\mathbf{v}_i = [v_i(1)', v_i(2)', \ldots]'$*,* $i \in \mathbb{I}_M$ *and a constant* $\sigma_r > 0$ *exist such that*

$$\sum_{i=1}^{M} w_i \sum_{j=1}^{\infty} L_i(\xi_i(j), v_i(j)) \leq \sigma_r \|e_i(0)\|^2, \tag{26}$$

the perturbed closed-loop system

$$\widehat{x}_i(k+1) = A_i \widehat{x}_i(k) + B_i u_i^p(\widehat{\mu}(k), 0) + \sum_{l \neq i}^{M} W_{il} u_l^p(\widehat{\mu}(k), 0) + Z_i e_i(k),$$
$$e_i(k+1) = \mathcal{A}_i^L e_i(k), \qquad i \in \mathbb{I}_M,$$

is exponentially stable for all $((\widehat{x}_i(0), e_i(0)),\ i \in \mathbb{I}_M) \in \mathbb{Z}$ *(Equation (25)).*

Proof. To establish exponential stability, we choose a candidate Lyapunov function that combines the regulator cost function and the subsystem state estimation errors ([5] employ a similar idea to show exponential stability of a single (centralized) MPC under output feedback). Define $V_N^p((\widehat{x}_i, e_i),\ i \in \mathbb{I}_M) = J_N^p(\widehat{\mu}) + \frac{1}{2}\sum_{i=1}^{M} w_i e_i' \Psi_i e_i$ to be a candidate Lyapunov function, in which Ψ_i is the solution of the Lyapunov equation $\mathcal{A}_i^{L\prime} \Psi_i \mathcal{A}_i^L - \Psi_i = -\Pi_i$ and $\Pi_i > 0$ is a user-defined matrix. Since $\mathcal{A}_i^L$ is a stable matrix and $\Pi_i > 0$, it follows that Ψ_i exists, is unique and positive definite (p.d.) [33, p. 230] for all $i \in \mathbb{I}_M$. Consider any $\widehat{x}_i(0) = \widehat{x}_i$ and $e_i(0) = e_i,\ i \in \mathbb{I}_M$ such that $((\widehat{x}_i, e_i),\ i \in \mathbb{I}_M) \in \mathbb{Z}$. We need to show [39, p. 267] that there exists constants $a, b, c > 0$ such that

$$a \sum_{i=1}^{M} \left[\|\widehat{x}_i\|^2 + \|e_i\|^2\right] \leq V_N^p((\widehat{x}_i, e_i),\ i \in \mathbb{I}_M) \leq b \sum_{i=1}^{M} \left[\|\widehat{x}_i\|^2 + \|e_i\|^2\right] \tag{27a}$$

$$\Delta V_N((\widehat{x}_i, e_i),\ i \in \mathbb{I}_M) \leq -c \sum_{i=1}^{M} \left[\|\widehat{x}_i\|^2 + \|e_i\|^2\right], \tag{27b}$$

in which $\Delta V_N^p((\widehat{x}_i, e_i),\ i \in \mathbb{I}_M) = V_N^q\left((\widehat{x}_i^+, e_i^+),\ i \in \mathbb{I}_M\right) - V_N^p((\widehat{x}_i, e_i),\ i \in \mathbb{I}_M)$.

For subsystem $i \in \mathbb{I}_M$, let $\mathbf{x}_i^p = [\rho_i^p(1)', \rho_i^p(2)', \ldots]', \rho_i^p(1) = \widehat{x}_{i,\ominus}^+ = F_i^p(\widehat{x}_i)$ be the state trajectory generated by the input trajectories $\mathbf{u}_1^p, \ldots, \mathbf{u}_M^p$, obtained after p Algorithm 1 iterates, and initial subsystem state $\widehat{x}_i$ (see Figure 7). Let $\mathbf{w}_i = \mathbf{u}_i^+(\mu) + \mathbf{v}_i,\ i \in \mathbb{I}_M$ (see Equation (5)) be a set of feasible subsystem input trajectories from $\widehat{\mu}^+$. The set of input trajectories $\mathbf{w}_1, \ldots, \mathbf{w}_M$ is used to initialize Algorithm 1 at the subsequent time step (from $\widehat{\mu}^+$). Let $\mathbf{z}_i = [z_i(2)', z_i(3)', \ldots]'$ denote the state trajectory generated by the set of feasible input trajectories $\mathbf{w}_1, \ldots, \mathbf{w}_M$, in which $z_i(2) = A_i \widehat{x}_i^+ + B_i w_i(1) + \sum_{j \neq i}^{M} w_j(1)$. For convenience, we define $z_i(1) = \widehat{x}_i^+,\ \forall\, i \in \mathbb{I}_M$. By definition (see p. 30), we have $z_i(j) = \rho_i^p(j) + \xi_i(j),\ j \geq 1,\ i \in \mathbb{I}_M$, and from Equation (19), $\xi_i(1) = Z_i e_i,\ i \in \mathbb{I}_M$. Using Lemma 4, we have

$$\begin{aligned} J_N^q(\widehat{\mu}^+) = \Phi([\mathbf{u}_1^q, \ldots, \mathbf{u}_M^q]; \widehat{\mu}^+) &\leq \Phi([\mathbf{w}_1, \ldots, \mathbf{w}_M]; \widehat{\mu}^+) \\ &= \sum_{i=1}^{M} w_i \sum_{j=1}^{\infty} L_i(z_i(j), w_i(j)) \\ &= \sum_{i=1}^{M} w_i \sum_{j=1}^{\infty} L_i(\rho_i^p(j) + \xi_i(j), u_i^p(j) + v_i(j)) \end{aligned}$$

Invoking Lemma 11 gives,

$$\leq \sum_{i=1}^{M} w_i \sum_{j=1}^{\infty} \left[(1+\delta) L_i(\rho_i^p(j), u_i^p(j)) + \left(1 + \frac{1}{\delta}\right) L(\xi_i(j), v_i(j)) \right]$$

Hence, we have

$$\begin{aligned} J_N^q(\widehat{\mu}^+) &\leq (1+\delta) \sum_{i=1}^{M} w_i \sum_{j=1}^{\infty} L_i(\rho_i^p(j), u_i^p(j)) + \left(1+\frac{1}{\delta}\right) \sum_{i=1}^{M} w_i \sum_{j=1}^{\infty} L_i(\xi_i(j), v_i(j)) \\ &\leq (1+\delta) \left[J_N^p(\widehat{\mu}) - \sum_{i=1}^{M} w_i L_i(\widehat{x}_i, u_i^p(\mu, 0)) \right] + \left(1+\frac{1}{\delta}\right) \sigma_r \sum_{i=1}^{M} \|e_i\|^2 \end{aligned}$$

We know $J_N^p(\mu) \leq b_r \sum_{i=1}^{M} \|x_i\|^2$. Let $\omega = \min_{i \in \mathbb{I}_M} w_i \frac{1}{2} \lambda_{\text{min}}(Q_i)$. Therefore,

$$\begin{aligned} J_N^q(\widehat{\mu}^+) - J_N^p(\mu) &\leq \delta J_N^p(\widehat{\mu}) - \sum_{i=1}^{M} w_i L_i(\widehat{x}_i, u_i^p(\mu, 0)) + \left(1+\frac{1}{\delta}\right) \sigma_r \sum_{i=1}^{M} \|e_i\|^2 \\ &\leq -(\omega - \delta b_r) \sum_{i=1}^{M} \|\widehat{x}_i\|^2 + \left(1+\frac{1}{\delta}\right) \sigma_r \sum_{i=1}^{M} \|e_i\|^2 \end{aligned}$$

Since $w_i, Q_i > 0, \; \forall\, i \in \mathbb{I}_M$, $\omega > 0$. Subsequently, we can choose $0 < c < \omega$ and $\delta_* = \dfrac{\omega - c}{b_r} > 0$. Let $d = \sigma_r \left(1 + \frac{1}{\delta_*}\right)$. We have

$$J_N^q(\widehat{\mu}^+) - J_N^p(\mu) \leq -c \sum_{i=1}^{M} \|x_i\|^2 + d \sum_{i=1}^{M} \|e_i\|^2$$

Define

$$\begin{aligned} \Delta e &= \frac{1}{2} \sum_{i=1}^{M} w_i {e_i^+}' \Psi_i e^+(k+1) - \frac{1}{2} \sum_{i=1}^{M} w_i {e_i}' \Psi_i e_i \\ &= \frac{1}{2} \sum_{i=1}^{M} w_i \left\{ {e_i}' \left[\mathcal{A}_i^{L\prime} \Psi_i \mathcal{A}_i^L - \Psi_i \right] e_i \right\} \\ &= -\frac{1}{2} \sum_{i=1}^{M} w_i {e_i}' \Pi_i e_i \end{aligned}$$

Let $w_{\min} = \min_{i \in \mathbb{I}_M} w_i$. The restriction $w_i > 0, i \in \mathbb{I}_M$ implies $w_{\min} > 0$. Since Π_i is a user-defined matrix, we can choose $\Pi_i = \Pi, \forall\, i \in \mathbb{I}_M$ such that $\lambda_{\text{min}}(\Pi) = \frac{2}{w_{\min}}(d+c)$ [3]. Noting that

[3] e.g., choose Π to be any diagonal matrix with the smallest diagonal entry equal to $\frac{2}{w_{\min}}(d+c)$.

$\Delta V_N(\cdot) = J_N^q(\widehat{\mu}^+) - J_N^p(\widehat{\mu}) + \Delta e$ gives

$$\begin{aligned}\Delta V_N\left((\widehat{x}_i, e_i),\ i \in \mathbb{I}_M\right) &\leq -c\sum_{i=1}^{M} \|\widehat{x}_i\|^2 + d\sum_{i=1}^{M}\|e_i\|^2 - \frac{1}{2}\left(\min_{i\in\mathbb{I}_M} w_i\lambda_{\min}(\Pi_i)\right)\sum_{i=1}^{M}\|e_i\|^2 \\ &= -c\sum_{i=1}^{M}\|\widehat{x}_i\|^2 - \left(\frac{1}{2}w_{\min}\lambda_{\min}(\Pi) - d\right)\sum_{i=1}^{M}\|e_i\|^2 \\ &= -c\sum_{i=1}^{M}\left[\|\widehat{x}_i\|^2 + \|e_i\|^2\right]\end{aligned}$$

Since $\Psi_i > 0,\ \forall\, i \in \mathbb{I}_M$, there exists constants $a_e, b_e > 0$ such that

$$a_e\sum_{i=1}^{M}\|e_i\|^2 \leq \frac{1}{2}\sum_{i=1}^{M} w_i e_i{}'\Psi_i e_i \leq b_e\sum_{i=1}^{M}\|e_i\|^2.$$

The choice $a = \min(a_r, a_e)$, $b = \max(b_r, b_e)$ satisfies Equation (27a). □

Proof for Theorem 1. From Lemma 2, we have (A_i, C_i) is detectable. It follows from Section 2 that there exists $\mathcal{L}_i, \forall\, i \in \mathbb{I}_M$ such that $\mathcal{A}_i^L = (A_i - A_i\mathcal{L}_iC_i)$ is a stable matrix. From [33, p. 231], $\overline{Q}_i > 0,\ i \in \mathbb{I}_M$. Following arguments used in the proof for [38, Theorem 1], constants a_r, b_r and c_r that satisfy Equation (24) can be determined. We note that $\mathbf{u}_i^+(\widehat{\mu}),\ i \in \mathbb{I}_M$ is a set of feasible input trajectories for the successor subsystem states $\widehat{\mu}^+$. For the choice $\mathbf{v}_i = [0, 0, \ldots]',\ \forall\, i \in \mathbb{I}_M$, Equation (22) gives $\xi_i(j) = A_i^{j-1}\xi_i(1), 1 \leq j$. Hence,

$$\begin{aligned}\sum_{i=1}^{M} w_i\sum_{j=1}^{\infty} L_i(\xi_i(j), v_i(j)) = \sum_{i=1}^{M} w_i\sum_{j=1}^{\infty} L_i(\xi_i(j), 0) &= \sum_{i=1}^{M} w_i\sum_{j=1}^{\infty}\xi_i(j)' A_i^{j-1\prime} Q_i A_i^{j-1}\xi_i(j) \\ &= \sum_{i=1}^{M} w_i\xi_i(1)'\overline{Q}_i\xi_i(1) \\ &\leq \sum_{i=1}^{M} w_i\lambda_{\max}(\overline{Q}_i)\|Z_i\|^2\|e_i\|^2\end{aligned}$$

The choice $\sigma_r = \max(w_1\lambda_{\max}(\overline{Q}_i)\|Z_i\|^2, \ldots, w_M\lambda_{\max}(\overline{Q}_M)\|Z_M\|^2)$ satisfies Equation (26). Invoking Theorem 5 with $\mathbb{Z} = \mathbb{R}^n \times \mathbb{R}^n$ completes the proof. □

Proof for Theorem 2. Existence of $\mathfrak{L}_i = \mathcal{L}_i, i \in \mathbb{I}_M$ such that $\mathcal{A}_i^L = (A_i - A_i\mathcal{L}_iC_i)$ is a stable matrix follows from Lemma 2. A procedure for determining constants a_r, b_r and c_r satisfying Equation (24) is given in the proof for [38, Theorem 2]. Consider Figure 7. We have using Equation (20) and the definition of $\mathbf{u}_i^p(\mu)$ that $U_{u_i}{}'\rho_i^p(N) = U_{u_i}{}'\rho_i^p(N+1) = 0$. For $((\widehat{x}_i, e_i),\ i \in \mathbb{I}_M) \in \mathbb{D}_C$ (see Section 3.1.2), $\overline{\mathbf{v}}_i, i \in \mathbb{I}_M$ exists. One possible choice for $\overline{\mathbf{v}}_i$ is the solution to the QP of Equation (8). Let $\mathbf{v}_i = [\overline{\mathbf{v}}_i{}', 0, \ldots]'$ and define $\mathbf{w}_i = \mathbf{u}_i^+(\widehat{\mu}) + \mathbf{v}_i,\ i \in \mathbb{I}_M$ to be admissible input trajectories from $\widehat{\mu}^+$ satisfying $U_{u_i}{}'z_i(N+1) = 0,\ i \in \mathbb{I}_M$. Hence, $U_{u_i}{}'\xi_i(N+1) = 0, \forall\, i \in \mathbb{I}_M$. Let $j \in \mathbb{I}_+ \cup \{0\}$. From Lemma 12, a constant K_{e_i} independent of e_i exists for each $i \in \mathbb{I}_M$ such that

$\|v_i(j)\| \leq K_{e_i}\|e_i\|,\ 0 \leq j$ and $U_{u_i}{}'\xi_i(N+1) = 0$. Let $\mathcal{A}_i = \max_{0\leq j\leq N} \|A_i^j\|$ and $\mathcal{A} = \max_{i\in\mathbb{I}_M} \mathcal{A}_i$. For each subsystem $i \in \mathbb{I}_M$, we have from Equation (22) that

$$\begin{aligned}
\|\xi_i(j)\| &\leq \|A_i^{j-1}\|\|\xi_i(1)\| + \sum_{l=1}^{j-1} \|A_i^{j-1-l}\|\|B_i\|\|v_i(l)\| + \sum_{l=1}^{j-1}\sum_{s\neq i}^{M} \|A_i^{j-1-l}\|\|W_{is}\|\|v_s(l)\| \\
&= \mathcal{A}\|Z_i\|\|e_i\| + \sum_{l=1}^{j-1} \mathcal{A}\|B_i\|K_{e_i}\|e_i\| + \sum_{s\neq i}^{M}\sum_{l=1}^{j-1} \mathcal{A}\|W_{is}\|K_{e_i}\|e_s\| \\
&\leq \mathcal{A}(\|Z_i\| + N\|B_i\|K_{e_i})\|e_i\| + \sum_{s\neq i}^{M} N\mathcal{A}\|W_{is}\|K_{e_s}\|e_s\| \\
&\leq \beta_{e_i}\sum_{s=1}^{M}\|e_s\|,\ \forall\, 1 \leq j \leq N+1,
\end{aligned}$$

in which $\beta_{e_i} = \max(\mathcal{A}(\|Z_i\| + N\|B_i\|K_{e_i}), \Xi_i)$ and $\Xi_i = N\mathcal{A}\max_{s\in\mathbb{I}_M}\|W_{is}\|K_{e_s}$. Let $\mathcal{F}^\infty = \sum_{i=1}^M w_i \sum_{j=1}^\infty L_i\left(\xi_i(j), v_i(j)\right)$. We have

$$\begin{aligned}
\mathcal{F}^\infty &= \sum_{i=1}^M w_i\left[\sum_{j=1}^N L_i(\xi_i(j), v_i(j)) + \sum_{j=N+1}^\infty L_i(\xi_i(j), v_i(j))\right] \\
&= \sum_{i=1}^M w_i\left[\sum_{j=1}^N L_i(\xi_i(j), v_i(j)) + \frac{1}{2}\xi_i(N+1)'\overline{Q}_i\xi_i(N+1)\right] \\
&\leq \sum_{i=1}^M w_i\frac{1}{2}\left\{\left[\lambda_{\max}(Q_i)N + \lambda_{\max}(\overline{Q}_i)\right]\beta_{e_i}^2\left(\sum_{s=1}^M\|e_s\|\right)^2 + N\lambda_{\max}(R_i)K_{e_i}^2\|e_i\|^2\right\}
\end{aligned}$$

Invoking Corollary 1.1 with $p, q = 2$ and $n = M$ gives, $\left(\sum_{i=1}^M\|e_i\|\right)^2 \leq M\sum_{i=1}^M\|e_i\|^2$. Hence,

$$\mathcal{F}^\infty \leq \sum_{i=1}^M w_i\frac{1}{2}\left\{\left[\lambda_{\max}(Q_i)N + \lambda_{\max}(\overline{Q}_i)\right]\beta_{e_i}^2 M\sum_{s=1}^M\|e_s\|^2 + N\lambda_{\max}(R_i)K_{e_i}^2\|e_i\|^2\right\}$$

Define the constants

$$\eta_a = \max_{i\in\mathbb{I}_M}\frac{1}{2}w_i\left[\lambda_{\max}(Q_i)N + \lambda_{\max}(\overline{Q}_i)\right]\beta_{e_i}^2 M, \quad \eta_b = \max_{i\in\mathbb{I}_M}\frac{1}{2}w_i N\lambda_{\max}(R_i)K_{e_i}^2$$

and $\eta = M\eta_a + \eta_b$. This gives $\mathcal{F}^\infty \leq \eta\sum_{i=1}^M\|e_i\|^2,\ \eta > 0$. Choosing $\sigma_r = \eta$ and invoking Theorem 5 with $\mathbb{Z} = \mathbb{D}_C$ proves the theorem.

□

C Disturbance modeling and distributed target calculation for FC-MPC

Proof for Lemma 6. From the Hautus Lemma for detectability [33, p. 318], $(\widetilde{A}_i, \widetilde{C}_i)$ is detectable iff $\text{rank}\left(\begin{bmatrix} \lambda I - \widetilde{A}_i \\ \widetilde{C}_i \end{bmatrix}\right) = n_i + n_{d_i}, \ \forall\, |\lambda| \geq 1$. Define

$$\mathcal{S}(\lambda) = \begin{bmatrix} \lambda I - A_{ii} & -\mathcal{B}^d_{ii} & \\ & (\lambda-1)I & \\ C_{ii} & \mathcal{C}^d_i & \mathbb{C}_s \\ & & \lambda I - \mathbb{A}_s \end{bmatrix} = \begin{bmatrix} \lambda I - \widetilde{A}_{ii} & 0 \\ \widetilde{C}_{ii} & \mathbb{C}_s \\ 0 & \lambda I - \mathbb{A}_s \end{bmatrix},$$

in which $\widetilde{A}_{ii} = \begin{bmatrix} A_{ii} & \mathcal{B}^d_{ii} \\ & I \end{bmatrix}$, $\widetilde{C}^d_{ii} = \begin{bmatrix} C_{ii} & \mathcal{C}^d_{ii} \end{bmatrix}$ denote respectively, the A and C matrix for the augmented decentralized model and

$$\begin{aligned} \mathbb{A}_s &= \text{diag}(A_{i1}, \ldots, A_{i(i-1)}, A_{i(i+1)}, \ldots, A_{iM}),\ A_{ii} \in \mathbb{R}^{n_{ii} \times n_{ii}} \\ \mathbb{C}_s &= [C_{i1}, \ldots, C_{i(i-1)}, C_{i(i+1)}, \ldots, C_{iM}] \end{aligned}$$

Consider $|\lambda| \geq 1$. Since $\mathbb{A}_s$ is stable, the columns of $\lambda I - \mathbb{A}_s$ are independent. Hence the columns of $\begin{bmatrix} 0 \\ \mathbb{C}_s \\ \lambda I - \mathbb{A}_s \end{bmatrix}$ are independent. By assumption, $\begin{bmatrix} \lambda I - \widetilde{A}_{ii} \\ \widetilde{C}_{ii} \end{bmatrix}$ has $n_{ii} + n_{d_i}$ independent columns. Due to the positions of the zeros, $\mathcal{S}(\lambda)$ has $n_i + n_{d_i}$ independent columns. To complete the proof, we note that

$$\begin{bmatrix} \lambda I - \widetilde{A}_i \\ \widetilde{C}_i \end{bmatrix} = U\mathcal{S}(\lambda)U,$$

in which U is an unitary matrix. Consequently, the columns of $\begin{bmatrix} \lambda I - \widetilde{A}_i \\ \widetilde{C}_i \end{bmatrix}$ are independent for all $|\lambda| \geq 1$. Hence, $(\widetilde{A}_i, \widetilde{C}_i)$ is detectable. □

Proof. $(\widetilde{A}_i, \widetilde{C}_i)$ **detectable** $\implies$ **rank condition.** From the Hautus lemma for detectability [33, p. 318], the columns of $\begin{bmatrix} \lambda I - A_i & -B^d_i \\ & (\lambda-1)I \\ C_i & C^d_i \end{bmatrix}$ are independent for any λ satisfying $|\lambda| \geq 1$. The columns of $\begin{bmatrix} \lambda I - A_i & -B^d_i \\ C_i & C^d_i \end{bmatrix}$ are, therefore, independent for all $|\lambda| \geq 1$. The choice $\lambda = 1$ gives the desired rank relationship.

rank condition $\implies$ $(\widetilde{A}_i, \widetilde{C}_i)$ **detectable.** The assumed rank condition implies the columns of $\begin{bmatrix} I - A_i & -B^d_i \\ C_i & C^d_i \end{bmatrix}$ are independent. From Lemma 2, (A_i, C_i) is detectable. The columns of $\begin{bmatrix} \lambda I - A_i \\ C_i \end{bmatrix}$ are, therefore, independent for all $|\lambda| \geq 1$. Consider $\begin{bmatrix} \lambda I - A_i & -B^d_i \\ C_i & C^d_i \\ 0 & (\lambda-1)I \end{bmatrix}$. For $\lambda \neq 1$, the columns of $(\lambda - 1)I$ are independent and therefore, the columns of $\begin{bmatrix} -B_i^d \\ C_i^d \\ (\lambda-1)I \end{bmatrix}$ are independent. Due to the position of the zero, the columns of $\begin{bmatrix} \lambda I - A_i & -B_i^d \\ C_i & C_i^d \\ 0 & (\lambda-1)I \end{bmatrix}$ are independent. For $\lambda = 1$, we know that the columns of $\begin{bmatrix} I - A_i & -B_i^d \\ C_i & C_i^d \end{bmatrix}$ are independent (by assumption). Hence, $(\widetilde{A}_i, \widetilde{C}_i)$ is detectable, as claimed. □

Theorem 6. *Let* $f(x) = \frac{1}{2}x'Qx + c'x + d$ *and* $-\infty < \underline{f} \leq f(x),\ \forall\, x$. *Consider the constrained QP*

$$\min_x f(x) \quad \textit{subject to} \quad Ax = b, x \in \mathbb{X}$$

in which $x \in \mathbb{R}^n$, $b \in \mathbb{R}^p$, $Q \geq 0$, $A \in \mathbb{R}^{p\times n}$, *and* $\mathbb{X} \subseteq \mathbb{R}^{s\times n}$ *is polygonal. Let the feasible region be nonempty. Let* $\text{rank}(A) = p$. *A solution to this problem exists. Furthermore, the solution is unique if rank* $\begin{bmatrix} Q \\ A \end{bmatrix} = n$.

Proof. Since the feasible region is nonempty and polygonal, and the QP is bounded below by $\underline{f}$, a solution exists [10]. Suppose that there exists two solutions x and $\overline{x}$. Let $w = x - \overline{x}$. We have $Aw = A(x - \overline{x}) = b - b = 0$. The normal cone optimality conditions [4] for x and $\overline{x}$ gives

$$\begin{aligned} (y - x)'(Qx + c) &\geq 0 \quad \forall\, y | Ay = b, y \in \mathbb{X} \\ (y - \overline{x})'(Q\overline{x} + c) &\geq 0 \quad \forall\, y | Ay = b, y \in \mathbb{X} \end{aligned}$$

Substituting $y = \overline{x}$ in the first equation and $y = x$ in the second equation, we have $w'Qx \leq -w'c$ and $w'Q\overline{x} \geq -w'c$. These two equations together imply $w'Q\overline{x} \geq w'Qx$, and therefore $w'Qw \leq 0$. Because $Q \geq 0$, $w'Qw \geq 0$. Hence, $w'Qw = 0$, which implies $Qw = 0$. Using $Aw = 0$ and full column rank for $\begin{bmatrix} Q \\ A \end{bmatrix}$, we have that the only solution for $\begin{bmatrix} Q \\ A \end{bmatrix} w = 0$ is $w = 0$. This gives $x = \overline{x}$. □

Proof for Lemma 9. Reverse implication. The objective function for the optimization problem of Equation (12) can be rewritten as $\Psi_i(\cdot) = \frac{1}{2} \begin{pmatrix} x_{s_{ii}} \\ u_{s_i} \end{pmatrix}' \begin{bmatrix} 0 & \\ & R_{u_i} \end{bmatrix} \begin{pmatrix} x_{s_{ii}} \\ u_{s_i} \end{pmatrix} + \begin{bmatrix} 0 \\ -R_{u_i} u_i^{\text{ss}} \end{bmatrix}' \begin{pmatrix} x_{s_{ii}} \\ u_{s_i} \end{pmatrix} + \frac{1}{2} u_i^{\text{ss}\prime} R_{u_i} u_i^{\text{ss}}$. From Theorem 6, the solution to the target optimization problem for each $i \in \mathbb{I}_M$ is unique if the columns of $\begin{bmatrix} 0 & \\ & R_{u_i} \\ I - A_{ii} & -B_{ii} \\ H_i C_{ii} & \end{bmatrix}$, $i \in \mathbb{I}_M$ are independent. Because $R_{u_i} > 0, i \in \mathbb{I}_M$, and

[4] We note that $f(\cdot)$ is a proper convex function in the sense of [28, p. 24], that the relative interior of $Ax = b, x \in \mathbb{X}$ is nonempty and that the feasible region defined by $(Ax = b, x \in \mathbb{X}) \subset \text{dom}(f(\cdot))$. The normal cone optimality conditions are, therefore, both necessary and sufficient [28, Theorem 27.4, p. 270].

due to the position of the zeros, the columns of $\begin{bmatrix} 0 & \\ & R_{u_i} \\ I - A_{ii} & -B_{ii} \\ H_i C_{ii} & \end{bmatrix}$ are independent if and only if the columns of $\begin{bmatrix} I - A_{ii} \\ H_i C_{ii} \end{bmatrix}$ are independent.

Forward implication. Let $(x_{s_{ii}}^{*(t)}, u_{s_i}^{*(t)}),\ i \in \mathbb{I}_M$ be unique and assume $\text{rank} \begin{bmatrix} I - A_{ii} \\ H_i C_{ii} \end{bmatrix} < n_{ii}$ for some $i \in \mathbb{I}_M$. By assumption, there exists v such that $\begin{bmatrix} I - A_{ii} \\ H_i C_{ii} \end{bmatrix} v = 0,\ v \neq 0$. The pair $(x_{s_{ii}}^{*(t)} + v, u_{s_i}^{*(t)})$ achieves the optimal cost $\frac{1}{2}\|u_i^{\text{ss}} - u_{s_i}^{*(t)}\|_{R_{u_i}}^2$ and

$$\begin{bmatrix} I - A_{ii} \\ H_i C_{ii} \end{bmatrix} (x_{s_{ii}}^{*(t)} + v) = \begin{bmatrix} B_{ii} u_{s_i}^{*(t)} + B_{ii}^d \widehat{d}_i \\ z_i^{\text{sp}} - H_i C_i^d \widehat{d}_i - \sum_{j \neq i}^M \left(\overline{g}_{ij} u_{s_j}^{t-1} + \overline{h}_{ij} d_i \right) \end{bmatrix},$$

which contradicts uniqueness of $x_{s_{ii}}^{*(t)}$. □

Proof for Theorem 3. Since $(\widetilde{A}_i, \widetilde{C}_i),\ i \in \mathbb{I}_M$ is detectable, an estimator gain $\widetilde{\mathcal{L}}_i$ exists such that $(\widetilde{A}_i - \widetilde{A}_i \widetilde{\mathcal{L}}_i \widetilde{C}_i)$ is stable for each $i \in \mathbb{I}_M$. From the positive invariance of $\widetilde{\mathbb{D}}_C$, $\left((z_i^{\text{sp}}, \widehat{d}_i(k)),\ i \in \mathbb{I}_M\right) \in \mathbb{D}_T$ for all $k \geq 0$. The target optimization problem (Equation (12)) is feasible for each $i \in \mathbb{I}_M$ for all $k \geq 0$. In Theorem 5 (Appendix B.2), replace e_i by $\widetilde{e}_i$, Z_i by $\tau_i^x \widetilde{Z}_i$, $\mathcal{A}_i^L$ by $(\widetilde{A}_i - \widetilde{A}_i \widetilde{\mathcal{L}}_i \widetilde{C}_i)$, $\widehat{x}_i$ by $\widehat{\omega}_i$, and $\widehat{\mu}$ by $\widehat{\mu} - \mu_s^t$. The model matrices $(A_i, B_i, \{W_{ij}\}_{j \neq i}, C_i)$ are unaltered. From the definition of $\mathbb{D}_T$, and positive invariance of $\widetilde{\mathbb{D}}_C$, feasible perturbation trajectories $\mathbf{v}_i, i \in \mathbb{I}_M$ exist such that $v_i(j) + (u_i^p(\widehat{\mu} - \mu_s^t, j) + u_{s_i}^t) \in \Omega_i, j \geq 1, i \in \mathbb{I}_M$. Existence of σ_r for stable and unstable systems can be demonstrated using arguments used to prove Theorem 1 and Theorem 2, respectively (with appropriate variable name changes as outlined above). Invoking Theorem 5 completes the proof. □

Proof for Lemma 10. From Lemma 2, (A_i, C_i) is detectable and (A_i, B_i) is stabilizable. Zero offset steady-state tracking performance can be established in the FC-MPC framework through an extension of either [20, Theorem 4] or [23, Theorem 1]. Let Algorithm 1 be terminated after $p \in \mathbb{I}_+,\ p < \infty$ iterates. At steady state, using Lemma 4 we have $u_i^p(\mu(\infty), 0) = u_i^\infty(\mu(\infty), 0),\ i \in \mathbb{I}_M,\ p \in \mathbb{I}_+$. Let the targets generated by Algorithm 2 at steady state be $(x_{s_i}^\infty, u_{s_i}^\infty), \forall\, i \in \mathbb{I}_M$ (see Section 4.3). Let $(\widehat{x}_i(\infty), \widehat{d}_i(\infty))$ denote an estimate of the subsystem state and integrating disturbance vectors at steady state. From Equation (10), we have

$$\begin{aligned} \widehat{x}_i(\infty) &= A_i \widehat{x}_i(\infty) + B_i u_i^\infty(\mu(\infty), 0) + \sum_{j \neq i} W_{ij} u_j^\infty(\mu(\infty), 0) + B_{\text{d}_i} \widehat{d}_i(\infty) \\ &\quad + \mathcal{L}_{\text{x}_i} \left(y_i(\infty) - C_i \widehat{x}_i(\infty) - C_i^d \widehat{d}_i(\infty) \right) \\ \text{and} \qquad \widehat{d}_i(\infty) &= \widehat{d}_i(\infty) + \mathcal{L}_{\text{d}_i} \left(y_i(\infty) - C_i \widehat{x}_i(\infty) - C_i^d \widehat{d}_i(\infty) \right) \end{aligned}$$

Invoking [23, Lemma 3] for each subsystem $i \in \mathbb{I}_M$ gives $\mathcal{L}_{\text{d}_i}$ is full rank. Hence, $y_i(\infty) = C_i \widehat{x}_i(\infty) + C_i^d \widehat{d}_i(\infty)$ and $(\widehat{x}_i(\infty) - x_{s_i}^\infty) = A_i(\widehat{x}_i(\infty) - x_{s_i}^\infty) + B_i(u_i^\infty(\mu(\infty), 0) - u_{s_i}^\infty) + \sum_{j \neq i} W_{ij}(u_j^\infty(\mu(\infty), 0) -$

$u_{s_j}^{\infty}),\ i \in \mathbb{I}_M$. Because all input constraints are inactive at steady state, there exists $\mathcal{K}$ such that the solution to Algorithm 1 at steady state is

$$\begin{bmatrix} (u_1^{\infty}(\mu(\infty), 0) - u_{s_1}^{\infty}) \\ (u_2^{\infty}(\mu(\infty), 0) - u_{s_2}^{\infty}) \\ \vdots \\ (u_M^{\infty}(\mu(\infty), 0) - u_{s_M}^{\infty}) \end{bmatrix} = -\mathcal{K} \begin{bmatrix} (\widehat{x}_1(\infty) - x_{s_1}^{\infty}) \\ (\widehat{x}_2(\infty) - x_{s_2}^{\infty}) \\ \vdots \\ (\widehat{x}_M(\infty) - x_{s_M}^{\infty}) \end{bmatrix}$$

Stability of the closed-loop system requires $A_{\mathrm{CM}} - B_{\mathrm{CM}}\mathcal{K}$ to be a stable matrix. Therefore,

$$(I - A_{\mathrm{CM}} - B_{\mathrm{CM}}\mathcal{K}) \begin{bmatrix} (\widehat{x}_1(\infty) - x_{s_1}^{\infty}) \\ (\widehat{x}_2(\infty) - x_{s_2}^{\infty}) \\ \vdots \\ (\widehat{x}_M(\infty) - x_{s_M}^{\infty}) \end{bmatrix} = 0,$$

which gives $(\widehat{x}_i(\infty) - x_{s_i}^{\infty}) = 0,\ i \in \mathbb{I}_M$ and $u_i^{\infty}(\mu(\infty), 0) = u_{s_i}^{\infty}$. This implies

$$\begin{aligned} H_i y_i(\infty) - z_i^{\mathrm{sp}} &= \Big(H_i C_i \widehat{x}_i(\infty) + H_i C_i^d \widehat{d}_i(\infty)\Big) - \Big(H_i C_i x_{s_i}^{\infty} + H_i C_i^d \widehat{d}_i(\infty)\Big) \\ &= H_i C_i (\widehat{x}_i(\infty) - x_{s_i}^{\infty}) \\ &= 0 \end{aligned}$$

□