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Stability and optimality of distributed, linear model predictive control*
Part I: state feedback

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Abstract

A new framework is presented for distributed, linear model predictive control (MPC) with guaranteed nominal stability and performance properties. We first show that modeling the interactions between subsystems and exchanging trajectory information among MPCs (communication) is insufficient to provide even closed-loop stability. We next propose a cooperative distributed MPC framework, in which the objective functions of the local MPCs are modified to achieve systemwide control objectives. This approach allows practitioners to tackle large, interacting systems by building on the local MPC systems already in place. The iterations generated by the proposed distributed MPC algorithm are systemwide feasible, and the controller based on any intermediate termination of the algorithm provides nominal closed-loop stability and zero steady-state offset. If iterated to convergence, the distributed MPC algorithm achieves optimal, centralized MPC control. Three examples are presented to illustrate the benefits of the proposed distributed MPC framework.

Keywords: Distributed control, plantwide control, distributed MPC

1 Introduction

Over the last decade, model predictive control (MPC) has established itself as a premier advanced control technology, with extensive applications in the chemical industry sector that exploit some of the latest theoretical developments in the area [8; 28; 40]. Large-scale applications are invariably approached by first breaking them into smaller, more manageable subsystems. Local models and objectives are selected for these smaller subsystem controllers, and the interactions between the subsystems are ignored at the design stage. If the interactions are not severe, the inherent feedback provided by the subsystem controllers is often adequate to provide acceptable overall system performance. But it is well known that this decentralized control approach can result in unacceptable closed-loop behavior when the subsystems interact significantly or unexpectedly. An obvious alternative is to attempt a centralized design of the entire large-scale system. This centralized approach to large-scale systems is viewed by most practitioners as monolithic and inflexible. For most networked systems, the primary hurdles to centralized control are not computational but organizational. In many applications, plants are already operating with decentralized MPCs in place. Operators do not wish to invest in a complete control system redesign as required to implement centralized MPC. In some cases, different parts of the networked system are owned by different organizations making the comprehensive model development and maintenance of centralized control impractical. Unless these organizational impediments change in the future, centralized control of large, networked systems is useful primarily as a benchmark against which other control strategies can be compared and assessed.

The benefits and requirements for cross-integration of subsystem MPCs has been discussed in [15; 22]. A two level decomposition-coordination strategy for generalized predictive control, based on the master-slave paradigm was proposed in [20]. A plantwide control strategy that involves the integration of linear and nonlinear MPC has been described in [41; 42]. A distributed MPC framework, for control of systems in which the dynamics of each of the subsystems are independent (decoupled) but the local state and control variables of the subsystems are nonseparably coupled in the cost function, was proposed in [21]. In the distributed MPC framework described in [21], each subsystem's MPC computes optimal input trajectories for itself and for all its neighbors. A sufficient condition for stability has also been established. Ensuring the stability condition in [21] is satisfied is, however, a nontrivial exercise. Furthermore, as noted by the authors, the stability condition has some undesirable consequences: (i) Satisfaction of the stability condition requires increasing information exchange rates as the system approaches equilibrium; this information exchange requirement to preserve nominal stability is counter-intuitive. (ii) Increasing the prediction horizon may lead to instability due to violation of the stability condition; closed-loop performance deteriorates after a certain horizon length. A globally feasible, continuous time distributed MPC framework for multi-vehicle formation stabilization was proposed in [11]. In this problem, the subsystem dynamics are decoupled but the states are nonseparably coupled in the cost function. Stability is assured through the use of a compatibility constraint that forces the assumed and actual subsystem responses to be within a pre-specified bound of each other. The compatibility constraint introduces a fair degree of conservatism and may lead to performance that is quite different from the optimal, centralized MPC performance. Relaxing the compatibility constraint leads to an increase in the frequency of information exchange among subsystems required to ensure stability. The authors result that each subsystem's MPC needs to communicate only with its neighbors is a direct consequence of the assumptions made - the subsystem dynamics are decoupled and only the states of the neighbors affect the local subsystem stage cost. A decentralized MPC algorithm for systems in which the subsystem dynamics and cost function are independent of the influence of other subsystem variables but have coupling constraints that link the state and input variables of different subsystems has been proposed in [30]. Robust feasibility is established when the disturbances are assumed to be independent, bounded and a fixed, sequential ordering for the subsystems' MPC optimizations is allowed.

A distributed MPC algorithm for unconstrained, linear time-invariant (LTI) systems in which the dynamics of the subsystems are influenced by the states of interacting subsystems has been described in [4; 18]. A contractive state constraint is employed in each subsystem's MPC optimization and asymptotic stability is guaranteed if the system satisfies a matrix stability condition. An algorithmic framework for partitioning a plant into suitably sized subsystems for distributed MPC has been described in [25]. An unconstrained, distributed MPC algorithm for LTI systems is also described. However, convergence, optimality and closed-loop stability properties, for the distributed MPC framework described in [25], have not been established. A distributed MPC strategy, in which the effects of the interacting subsystems are treated as bounded uncertainties, has been described in [19]. Each subsystem's MPC solves a min-max optimization problem to determine local control policies. The authors show feasibility of their distributed MPC formulation; optimality and closed-loop stability properties are, however, unclear. Recently in [10], an extension of the distributed MPC framework described in [11] that handles systems with interacting subsystem dynamics was proposed. At each time step, existence of a feasible input trajectory is assumed for each subsystem. This assumption is one limitation of the formulation. Furthermore, the analysis in [10] requires at least 10 agents for closed-loop stability. This lower bound on the number of agents (MPCs) is an undesirable and artificial restriction and limits the applicability of the method.

To arrive at distributed MPC algorithms with guaranteed feasibility, stability and performance properties, we also examine contributions to the area of plantwide decentralized control. A survey of decentralized control methods for large-scale systems can be found in [33]. Performance limitations arising due to the decentralized control framework has been described in [7]. Several decentralized controller design approaches approximate or ignore the interactions between the various subsystems and lead to a suboptimal plantwide control strategy [1; 23; 32; 34]. The required characteristics of any problem solving architecture in which the agents are autonomous and influence one another's solutions has been described in [37].

This work provides a new approach for controlling large, networked systems through the suitable integration of subsystem-based MPCs. It is assumed that the interactions between the subsystems are stable; system redesign is recommended otherwise. The proposed cooperation-based distributed MPC algorithm is iterative with the subsystem-based MPC optimizations executed in parallel. The term *iterate* indicates a set of MPC optimizations executed in parallel (one for each subsystem) followed by an exchange of information among interconnected subsystems. The distributed MPC algorithm can be terminated at any intermediate iterate to allow for computational or communication limits. This approach is aimed at allowing practitioners to build on existing infrastructure. In many applications, the subsystems' MPCs are in place and running successfully. The proposed method provides practitioners with a low-risk strategy to explore the benefits achievable with centralized control by implementing cooperation-based MPCs instead.

The paper is organized as follows. In Section 2, the modeling framework for distributed MPC is described. Some basic definitions and results are presented in Section 3. The different candidate MPC formulations for systemwide control are described in Section 4. In Section 5, optimality conditions for distributed MPC are characterized. An implementable algorithm for distributed MPC is described in Section 6. Properties of the proposed distributed MPC algorithm are established subsequently. Closed-loop properties for the proposed distributed MPC framework under state feedback are established in Section 7. Three examples are presented in Section 8 to illustrate the efficacy of the proposed approach. Finally, in Section 9, a summary of the contributions of this study is provided.

2 Interaction modeling

Consider a plant comprising of M subsystems. The symbol $\mathbb{I}_M$ denotes the set of integers $1, 2, \ldots, M$.

Decentralized models. Let the decentralized (local) model for each subsystem $i \in \mathbb{I}_M$ be represented by a discrete, linear time invariant (LTI) model of the form

$$\begin{aligned} x_{ii}(k+1) &= A_{ii}x_{ii}(k) + B_{ii}u_i(k) \\ y_i(k) &= C_{ii}x_{ii}(k) \end{aligned}$$

in which k is discrete time, and we assume $(A_{ii} \in \mathbb{R}^{n_{ii} \times n_{ii}}, B_{ii} \in \mathbb{R}^{n_{ii} \times m_i}, C_{ii} \in \mathbb{R}^{z_i \times n_{ii}})$ is a realization for each (u_i, y_i) input-output pair such that (A_{ii}, B_{ii}) is stabilizable and (A_{ii}, C_{ii}) is detectable.

Interaction models (IM). Consider any subsystem $i \in \mathbb{I}_M$. We represent the effect of any interacting subsystem $j \neq i$ on subsystem i through a discrete LTI model of the form

$$x_{ij}(k+1) = A_{ij}x_{ij}(k) + B_{ij}u_j(k)$$

The output equation for each subsystem is written as

$$y_i(k) = \sum_{j=1}^{M} C_{ij}x_{ij}(k)$$

$(A_{ij} \in \mathbb{R}^{n_{ij} \times n_{ij}}, B_{ij} \in \mathbb{R}^{n_{ij} \times m_j}, C_{ij} \in \mathbb{R}^{z_i \times n_{ij}})$ is a minimal realization of the input-output pair (u_j, y_i).

Composite models (CM). The combination of the decentralized model and the interaction models for each subsystem yields the composite model (CM). The decentralized state vector x_{ii} is augmented with states arising due to the effects of all other subsystems.

Let $x_i{}' = [x_{i1}{}', \ldots, x_{ii}{}', \ldots, x_{iM}{}']$ denote the CM states for subsystem i. For notational simplicity, we represent the CM for subsystem i as

$$x_i(k+1) = A_i x_i(k) + B_i u_i(k) + \sum_{j \neq i} W_{ij} u_j(k) \tag{1a}$$

$$y_i(k) = C_i x_i(k) \tag{1b}$$

in which $C_i = [C_{i1} \ldots C_{ii} \ldots C_{iM}]$ and

$$A_i = \begin{bmatrix} A_{i1} & & & & \\ & \ddots & & & \\ & & A_{ii} & & \\ & & & \ddots & \\ & & & & A_{iM} \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ \vdots \\ B_{ii} \\ 0 \\ \vdots \end{bmatrix}, \quad W_{ij} = \begin{pmatrix} 0 \\ \vdots \\ B_{ij} \\ 0 \\ \vdots \end{pmatrix}$$

The composite model (CM) for the entire plant can be written as

$$\left[\begin{array}{c} x_{11} \\ \vdots \\ x_{1M} \\ \hline \vdots \\ \hline x_{M1} \\ \vdots \\ x_{MM} \end{array}\right](k+1) = \underbrace{\left[\begin{array}{ccc|c|ccc} A_{11} & & & & & & \\ & \ddots & & & & & \\ & & A_{1M} & & & & \\ \hline & & & \ddots & & & \\ \hline & & & & A_{M1} & & \\ & & & & & \ddots & \\ & & & & & & A_{MM} \end{array}\right]}_{A_{\text{CM}}} \left[\begin{array}{c} x_{11} \\ \vdots \\ x_{1M} \\ \hline \vdots \\ \hline x_{M1} \\ \vdots \\ x_{MM} \end{array}\right](k) + \underbrace{\left[\begin{array}{ccc} B_{11} & & \\ & \ddots & \\ & & B_{1M} \\ \hline & \vdots & \\ \hline B_{M1} & & \\ & \ddots & \\ & & B_{MM} \end{array}\right]}_{B_{\text{CM}}} \begin{bmatrix} u_1 \\ \vdots \\ u_M \end{bmatrix}(k)$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_M \end{bmatrix}(k) = \underbrace{\left[\begin{array}{ccc|c|ccc} C_{11} & \cdots & C_{1M} & & & & \\ & & & \ddots & & & \\ & & & & C_{M1} & \cdots & C_{MM} \end{array}\right]}_{C_{\text{CM}}} \left[\begin{array}{c} x_{11} \\ \vdots \\ x_{1M} \\ \hline \vdots \\ \hline x_{M1} \\ \vdots \\ x_{MM} \end{array}\right](k)$$

in which $A_{\text{CM}} \in \mathbb{R}^{n \times n}, B_{\text{CM}} \in \mathbb{R}^{n \times m}, C_{\text{CM}} \in \mathbb{R}^{n_y \times n}$ and $n = \sum_{i=1}^{M} n_i$, $m = \sum_{i=1}^{M} m_i$ and $n_y = \sum_{i=1}^{M} n_{y_i}$. It is assumed that $(A_{\text{CM}}, B_{\text{CM}})$ is stabilizable and $(A_{\text{CM}}, C_{\text{CM}})$ is detectable.

After identification of the significant interactions from closed-loop operating data, we expect that many of the interaction terms will be zero. In the decentralized model, all of the interaction terms are zero. More details on closed-loop identification procedures for distributed MPC can be found in [13].

Centralized model. The centralized model is represented as

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

3 Notation and Preliminaries

For any vector $x \in \mathbb{R}^n$, the notation $\|x\|_P$ denotes the P-weighted norm defined as $\|x\|_P = \sqrt{x'Px}$. Unless specified otherwise, the notation $\|x\|$ denotes the Euclidean norm of x *i.e.*, $P = I$. For any matrix P, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalue of P respectively. Following the vector norm definitions we define, for any matrix A, the corresponding matrix norm as

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

For the case of the Euclidean norm, $\|A\|_2 = \sqrt{\lambda_{\max}(A'A)}$. For notational simplicity, we use $\|A\|$ to denote the Euclidean norm, unless specified otherwise.

For any $i \in \mathbb{I}_M$ the notation $j \neq i$ indicates that j can take all values in $\mathbb{I}_M$ except $j = i$. Let $\mathbb{I}_+$ denote the set of positive integers. Given a bounded set Λ, the notation int(Λ) denotes the interior of the set. For any two vectors $r, s \in \mathbb{R}^n$, the notation $\langle r, s \rangle$ represents the inner product of the two vectors. For any arbitrary, finite set of vectors $a_1, a_2, \ldots, a_s$, define

$$\text{vec}(a_1, a_2, \ldots, a_s) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_s \end{bmatrix}$$

and for a finite set of (compatible) matrices $Y_j \in \mathbb{R}^{s_j \times r},\ j = 1, 2, \ldots, J,\ J \in \mathbb{I}_+$,

$$\text{vec}(Y_1, \ldots, Y_J) = \begin{bmatrix} Y_1{}' & \ldots & Y_J{}' \end{bmatrix}'.$$

Let the space of the non-negative reals be represented by $\mathbb{R}^+$. Denote a closed ball of radius $b \in \mathbb{R}^+$ centered at $a \in \mathbb{R}^n$ by $B_b(a)$

$$B_b(a) = \{x | \, \|x - a\| \leq b\}$$

A real $n \times n$ matrix has a **Schur decomposition** [12, p. 341]

$$A = \begin{bmatrix} U_s & U_u \end{bmatrix} \begin{bmatrix} A_s & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} U_s' \\ U_u' \end{bmatrix}$$

in which $U = [U_s \ U_u]$ is a real and orthogonal $n \times n$ matrix, the eigenvalues of A_s are strictly inside the unit circle, and the eigenvalues of A_u are on or outside the unit circle.

Lemma 1. *Let $\mathcal{A}x = b$ be a system of linear equations with $\mathcal{A} \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m,\ m < n$. Consider $\mathcal{X} \subset \mathbb{R}^n$ nonempty, compact, convex with $0 \in \text{int}(\mathcal{X})$. The set $\mathcal{B} \subseteq \text{range}(\mathcal{A})$ is defined as*

$$\mathcal{B} = \{b \mid \mathcal{A}x = b, x \in \mathcal{X}\}$$

For every $b \in \mathcal{B}$, $\exists \overline{x}(b)$ dependent on b and $K > 0$ independent of b such that $\mathcal{A}\overline{x}(b) = b, \overline{x}(b) \in \mathcal{X}$ and $\|\overline{x}(b)\| \leq K\|b\|$

A proof is given in Appendix A.

Let the current (discrete) time be k. For any subsystem $i \in \mathbb{I}_M$, let the predicted state and input at time instant $k + j,\ j \geq 0$, based on data at time k be denoted by $x_i(k + j|k) \in \mathbb{R}^{n_i}$ and $u_i(k + j|k) \in \mathbb{R}^{m_i}$ respectively. The stage cost is defined as

$$L_i(x, u) = \frac{1}{2}\left[x'Q_i x + u'R_i u\right] \tag{2}$$

in which $Q_i \geq 0$, $R_i > 0$. Denote a closed ball of radius $\varepsilon > 0$ centered at $a \in \mathbb{R}^n$ by $B_\varepsilon(a) = \{x | \, \|x - a\| \leq \varepsilon\}$. The notation $\mu(k)$ is used to denote the set of CM states $x_1(k), x_2(k), \ldots x_M(k)$ *i.e.,*

$$\mu(k) = [x_1(k), x_2(k), \ldots, x_M(k)].$$

With slight abuse of notation, we use $\mu(k) \in \mathcal{X}$ to denote $\text{vec}(\mu(k)) = \text{vec}(x_1(k), x_2(k), \ldots, x_M(k)) \in \mathcal{X}$. The norm operator for $\mu(k)$ is defined as

$$\|\mu(k)\| = \|\text{vec}(x_1(k), x_2(k), \ldots, x_M(k))\| = \sqrt{\sum_{i=1}^{M} \|x_i(k)\|^2}.$$

We have the following definitions for the predicted infinite horizon state and input trajectory vectors in the different MPC frameworks.

$$\begin{aligned}
\text{Centralized state trajectory: } \mathbf{x}(k)' &= \left[x(k+1|k)', x(k+2|k)', \ldots\ldots\right] \\
\text{Centralized input trajectory: } \mathbf{u}(k)' &= \left[u(k|k)', u(k+1|k)', \ldots\ldots\right] \\
\text{CM state trajectory (subsystem } i\text{): } \mathbf{x}_i(k)' &= \left[x_i(k+1|k)', x_i(k+2|k)', \ldots\ldots\right] \\
\text{Input trajectory (subsystem } i\text{): } \mathbf{u}_i(k)' &= \left[u_i(k|k)', u_i(k+1|k)', \ldots\ldots\right] \\
\text{Decentralized state trajectory (subsystem } i\text{): } \mathbf{x}_{ii}(k)' &= \left[x_{ii}(k+1|k)', x_{ii}(k+2|k)', \ldots\ldots\right]
\end{aligned}$$

Let N denote the control horizon. The following notation is used to represent the finite horizon predicted state and input trajectory vectors in the different MPC frameworks

$$\begin{aligned}
\text{Centralized state trajectory: } \overline{\mathbf{x}}(k)' &= \left[x(k+1|k)', \ldots, x(k+N|k)'\right] \\
\text{Centralized input trajectory: } \overline{\mathbf{u}}(k)' &= \left[u(k|k)', \ldots, u(k+N-1|k)'\right] \\
\text{CM state trajectory (subsystem } i\text{): } \overline{\mathbf{x}}_i(k)' &= \left[x_i(k+1|k)', \ldots, x_i(k+N|k)'\right] \\
\text{Input trajectory (subsystem } i\text{): } \overline{\mathbf{u}}_i(k)' &= \left[u_i(k|k)', \ldots, u_i(k+N-1|k)'\right] \\
\text{Decentralized state trajectory (subsystem } i\text{): } \overline{\mathbf{x}}_{ii}(k)' &= \left[x_{ii}(k+1|k)', \ldots, x_{ii}(k+N|k)'\right]
\end{aligned}$$

Define

$$\mathcal{C}_N(A, B) = \begin{bmatrix} B & AB & \ldots & A^{N-1}B \end{bmatrix}.$$

Assumptions 1. All interaction models are stable *i.e.*, for each $i, j \in \mathbb{I}_M$, $|\lambda_{\max}(A_{ij})| < 1, \forall j \neq i$.

4 Systemwide control with MPC: problem formulation and assumptions

In this section, four MPC based systemwide control formulations are described. In each case, the controller is defined by implementing the first input in the solution to the corresponding optimization problem. Let $\Omega_i \subset \mathbb{R}^{m_i}$, the set of admissible controls for subsystem i, be a nonempty, compact, convex set containing the origin in its interior. The set of admissible controls for the whole plant Ω is the Cartesian product of the admissible control sets of each of the subsystems. It follows that Ω is a compact, convex set containing the origin in its interior. The constrained stabilizable set $\mathbb{X}$ is the set of all initial subsystem states $\mu = [x_1, x_2, \ldots, x_M]$ that can be steered to the origin by applying a sequence of admissible controls (see [36, Definition 2]). In each MPC based framework, $\mu(0) \in \mathbb{X}$. Hence a feasible solution exists to the corresponding optimization problem.

$$\mathcal{P}_1: \quad \textbf{Centralized MPC}$$

$$\min_{\mathbf{x}(k),\mathbf{u}(k)} \quad \phi\left(\mathbf{x}(k),\mathbf{u}(k);x(k)\right) = \sum_i w_i \phi_i\left(\mathbf{x}_i(k),\mathbf{u}_i(k);x_i(k)\right)$$

$$\begin{aligned} &\text{subject to} \\ &x(l+1|k) = Ax(l|k) + Bu(l|k), \quad k \leq l \\ &u_i(l|k) \in \Omega_i, \quad k \leq l, \forall\ i \in \mathbb{I}_M \\ &x(k) = \widehat{x}(k) \end{aligned}$$

in which $\mathbf{x}(k),\ \mathbf{u}(k)$ represents the centralized state and input trajectories. The cost function for subsystem i is ϕ_i. The system objective is a convex combination of the local objectives in which $w_i > 0, i \in \mathbb{I}_M,\ \sum_i w_i = 1$. The vector $\widehat{x}(k)$ represents the current estimate of the centralized model states $x(k)$ at discrete time k.

$$\mathcal{P}_2(i): \quad \textbf{Decentralized MPC}$$

$$\min_{\mathbf{x}_{ii}(k),\mathbf{u}_i(k)} \quad \phi_i^d\left(\mathbf{x}_{ii}(k),\mathbf{u}_i(k);x_{ii}(k)\right)$$

$$\begin{aligned} &\text{subject to} \\ &x_{ii}(l+1|k) = A_{ii}x_{ii}(l|k) + B_{ii}u_i(l|k), \quad k \leq l \\ &u_i(l|k) \in \Omega_i, k \leq l \\ &x_{ii}(k) = \widehat{x}_{ii}(k) \end{aligned}$$

in which $(\mathbf{x}_{ii}, \mathbf{u}_i)$ represents the decentralized state and input trajectories for subsystem $i \in \mathbb{I}_M$. The notation $\widehat{x}_{ii}(k)$ represents the estimate of the decentralized model states at discrete time k. The subsystem cost function in the decentralized MPC framework, $\phi_i^d\left(\mathbf{x}_{ii}(k),\mathbf{u}_i(k);x_{ii}(k)\right)$ is defined as

$$\phi_i^d\left(\mathbf{x}_{ii}(k),\mathbf{u}_i(k);x_{ii}(k)\right) = \frac{1}{2}\sum_{t=k}^{\infty}\left[x_{ii}(t|k)'Q_{ii}x_{ii}(t|k) + u_i(t|k)'R_iu_i(t|k)\right]$$

in which $Q_{ii} \geq 0$, $R_i > 0$ and $(A_{ii}, Q_{ii}^{1/2})$ is detectable.

For communication and cooperation-based MPC, an iteration and exchange of variables between subsystems is performed during a sample time. We may choose not to iterate to convergence. We denote this iteration number as p. The cost function for communication-based MPC is defined over an infinite horizon and written as

$$\phi_i\left(\mathbf{x}_i(k),\mathbf{u}_i(k);x_i(k)\right) = \sum_{t=k}^{\infty} L_i\left(x_i(t|k),u_i(t|k)\right) \tag{5}$$

(see Equation (2)), in which $Q_i \geq 0$, $R_i > 0$ are symmetric weighting matrices with $(A_i, Q_i^{1/2})$ detectable. For each subsystem i and iterate p, the optimal state-input trajectory $(\mathbf{x}_i^p(k), \mathbf{u}_i^p(k))$ is obtained as the solution to the optimization problem $\mathcal{P}_3(i)$ defined as

$$\mathcal{P}_3(i): \quad \textbf{Communication-based MPC}$$

$$\min_{\mathbf{x}_i^p(k), \mathbf{u}_i^p(k)} \quad \phi_i\left(\mathbf{x}_i^p(k), \mathbf{u}_i^p(k); x_i(k)\right)$$

subject to

$$x_i^p(l+1|k) = A_i x_i^p(l|k) + B_i u_i^p(l|k) + \sum_{j \neq i} W_{ij} u_j^{p-1}(l|k), \quad k \leq l$$

$$u_i^p(l|k) \in \Omega_i, \quad k \leq l$$

$$x_i(k) = \widehat{x}_i(k)$$

in which $\mathbf{x}_i^p(k)' = [x_i^p(k+1|k)', x_i^p(k+2|k)', \ldots, \ldots]$, $\mathbf{u}_i^p(k)' = [u_i^p(k|k)', u_i^p(k+1|k)', \ldots, \ldots]$ and $\widehat{x}_i(k)$ represents the current estimate of the composite model states. Notice that the input sequence for subsystem i, $\mathbf{u}_i^p(k)$, is optimized to produce its value at iteration p, but the other subsystems' inputs are not updated during this optimization; they remain at iterate $p-1$. The objective function is the one for subsystem i only. For notational simplicity, we drop the time dependence of the state and input trajectories in each of the MPC frameworks described above. For instance, we write $(\mathbf{x}_i^p, \mathbf{u}_i^p) \equiv (\mathbf{x}_i^p(k), \mathbf{u}_i^p(k))$.

Each communication-based MPC * transmits current state and input trajectory information to all interconnected subsystems' MPCs. Competing agents have no knowledge of each others cost/utility functions. From a game theoretic perspective, the equilibrium of such a strategy, if it exists, is called a noncooperative equilibrium or Nash equilibrium [2]. The objectives of each subsystem's MPC controller are frequently in conflict with the objectives of other interacting subsystems' controllers. The best achievable performance is characterized by a Pareto optimal path which represents the set of optimal trade-offs among these conflicting/competing controller objectives. It is well known that the Nash equilibrium is usually suboptimal in the Pareto sense [6; 9; 26].

4.1 Geometry of Communication-based MPC

We illustrate possible scenarios that can arise under communication-based MPC. In each case, $\Phi_i(\cdot)$ denotes the subsystem cost function obtained by eliminating the states from the cost function $\phi_i(\mathbf{x}_i, \mathbf{u}_i; x_i)$ using the subsystem CM equation (see p. 13). The Nash equilibrium (NE) and the Pareto optimal solution are denoted by n and p respectively. To allow a 2-dimensional representation, a unit control horizon ($N = 1$) is used. In each example, existence of the NE follows using [2, Theorem 4.4, p. 176]. The NE n is the point of intersection of the reaction curves of the two cost functions [2, p. 169]. The Pareto optimal path is the locus of (u_1, u_2) obtained by minimizing the weighted sum $w_1\Phi_1 +$

*Similar strategies have been proposed in [4; 18]

$w_2\Phi_2$ for each $0 \le w_1, w_2 \le 1, w_1 + w_2 = 1$. If $(w_1, w_2) = (1, 0)$, the Pareto optimal solution is at point a, and if $(w_1, w_2) = (0, 1)$, the Pareto optimal solution is at point b.

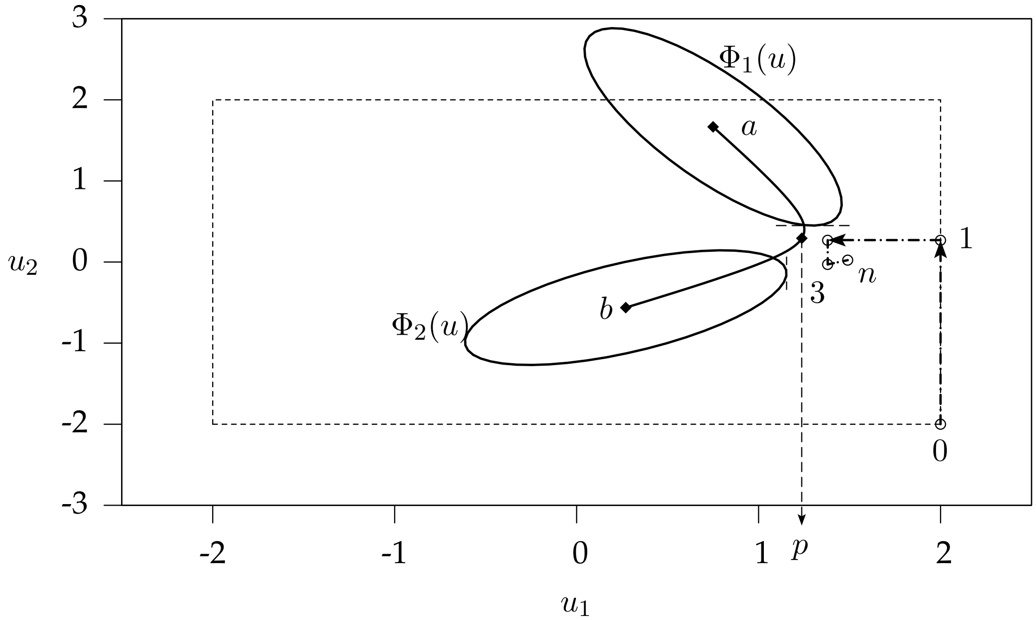

Figure 1: A stable Nash equilibrium exists and is near the Pareto optimal solution. Communication-based iterates converge to the stable Nash equilibrium.

Example 1. Figure 1 illustrates the best case scenario for pure communication strategies. The NE n is located near the Pareto optimal solution p. The box in Figure 1 represents constraints on u_1 and u_2. For initial values of u_1 and u_2 located at point 0, the first communication-based iterate steers u_1 and u_2 to point 1. On iterating further, the sequence of communication-based iterates converges to n. In this case, the NE is stable *i.e.*, if the system is displaced from n, the sequence of communication-based iterates brings the system back to n. The closed-loop system will likely behave well in this case.

Example 2. Here, the initial values of the inputs are located near the Pareto optimal solution (Point 0 in Figure 2). However, as observed from Figure 2, the NE n for this system is not near p and therefore, the sequence of communication-based iterates drives the system away from the Pareto optimal solution. Even though the NE is stable, the solution obtained at convergence (n) of the communication-based strategy is far from optimal. Consequently, a stable NE need not imply closed-loop stability.

Example 3. We note from Figure 3 that the NE (n) for this system is in the proximity of the Pareto optimal solution p. For initial values of u_1 and u_2 at the origin and in the absence of input constraints, the sequence of communication-based iterates diverges. For

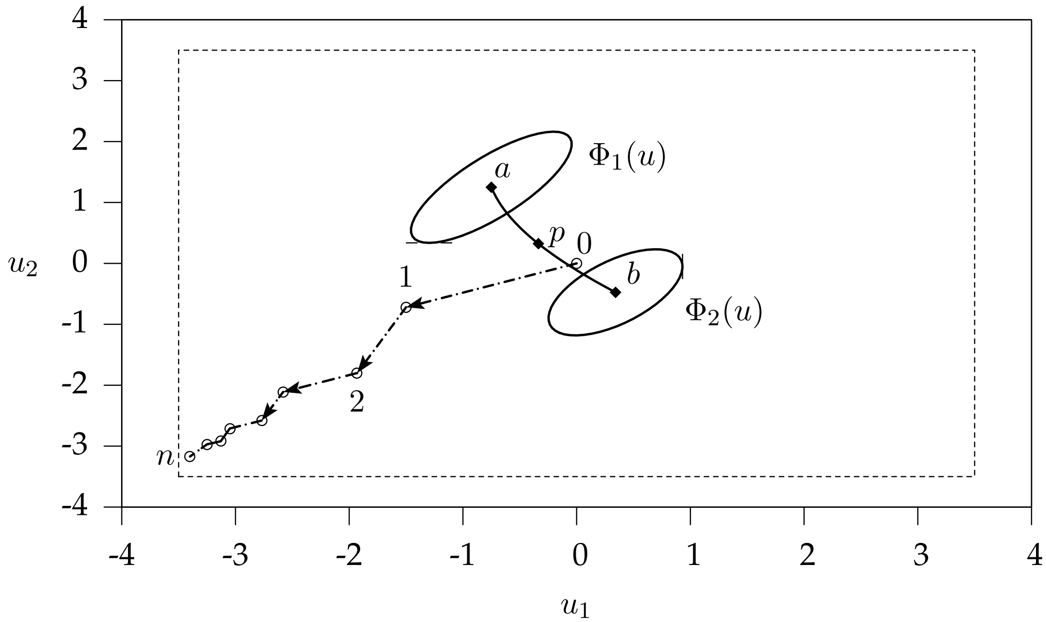

Figure 2: A stable Nash equilibrium exists but is not near the Pareto optimal solution. The converged solution, obtained using a communication-based strategy, is far from optimal.

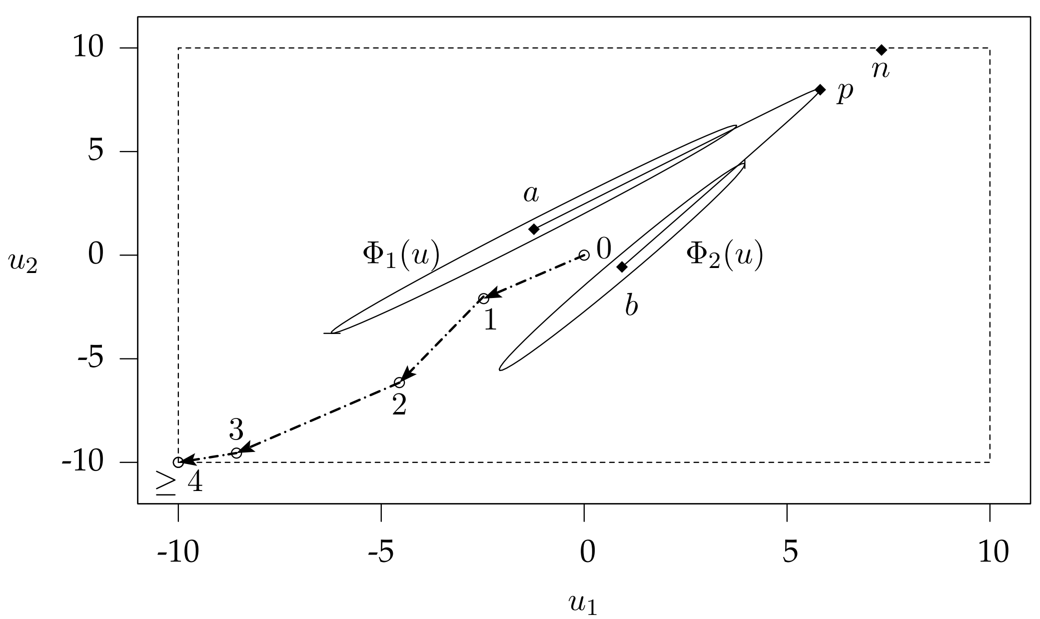

Figure 3: A stable Nash equilibrium does not exist. Communication-based iterates do not converge to the Nash equilibrium.

a compact feasible region (the box in Figure 3), the sequence of communication-based iterates is trapped at the boundary of the feasible region (Point 4) and does not converge to n. Here, a stable NE for a (pure) communication-based strategy, in the sense of [2, Definition 4.5, p. 172], does not exist. The closed-loop system is likely to be unstable in this case.

For strongly coupled systems, the NE may not be close to the Pareto optimal solution. In some situations (Example 3), communication-based strategies do not converge to the NE. In fact, it is possible to construct simple examples where communication-based MPC leads to closed-loop instability (Section 8). Communication-based MPC is therefore, an unreliable strategy for systemwide control. The unreliability of the communication-based MPC formulation as a systemwide control strategy motivates the need for an alternate approach. We next modify the objective functions of the subsystems' controllers in order to provide a means for cooperation among the controllers. We replace the objective ϕ_i with an objective that measures the systemwide impact of local control actions. Many suitable objectives are possible. Here we choose the simplest case, the overall plant objective, which is a strict convex combination of the individual subsystems' objectives, $\phi = \sum_i w_i \phi_i, \;\; w_i > 0, \;\; \sum_{i=1}^{M} w_i = 1$.

In practical situations, the process sampling interval may be insufficient for the computation time required for convergence of a cooperation-based iterative algorithm. In such situations, the cooperation-based distributed MPC algorithm has to be terminated prior to convergence of the state and input trajectories (*i.e.*, when time runs out). The last calculated input trajectory is used to arrive at a suitable control law. To allow intermediate termination, all iterates generated by the distributed MPC algorithm must be plantwide feasible, and the resulting controller must be closed-loop stable. By plantwide feasibility, we mean that the state–input sequence $\{\mathbf{x}_i, \mathbf{u}_i\}_{i=1}^{M}$ satisfies the model and input constraints of each subsystem. To guarantee plantwide feasibility of the intermediate iterates, we eliminate the states $\mathbf{x}_i, i \in \mathbb{I}_M$ from each of the optimization problems using the set of CM equations (Equation (1)). Subsequently, the cost function $\phi_i(\mathbf{x}_i, \mathbf{u}_i; x_i(k))$ can be re-written as a function of all the interacting subsystem input trajectories with the initial subsystem state as a parameter *i.e.*,

$$\phi_i(\mathbf{x}_i, \mathbf{u}_i; x_i(k)) \equiv \Phi_i\left([\mathbf{u}_1, \ldots, \mathbf{u}_i, \ldots, \mathbf{u}_M]; x_i(k)\right).$$

For each subsystem i, the optimal input trajectory $\mathbf{u}_i^{*(p)}$ is obtained as the solution to the feasible cooperation-based MPC (FC-MPC) optimization problem defined as

$$\mathcal{P}_4(i): \quad \textbf{Feasible cooperation-based MPC}$$

$$\mathbf{u}_i^{*(p)}(k) \in \;\; \arg(\mathcal{F}_i) \;\; \text{where}$$

$$\mathcal{F}_i \triangleq \min_{\mathbf{u}_i} \sum_{r=1}^{M} w_r \Phi_r \left([\mathbf{u}_1^{p-1}, \ldots, \mathbf{u}_{i-1}^{p-1}, \mathbf{u}_i, \mathbf{u}_{i+1}^{p-1}, \ldots, \mathbf{u}_M^{p-1}]; x_r(k) \right)$$

subject to

$$u_i(l|k) \in \Omega_i, \quad k \leq l$$

$$x_r(k) = \widehat{x}_r(k), \forall\, r \in \mathbb{I}_M$$

5 Distributed, constrained optimization

Consider the following prototype centralized MPC optimization problem.

$$\min_{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M} \Phi\left([\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M]; \mu(k)\right) = \sum_{i=1}^{M} w_i \Phi_i \left([\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M]; x_i(k)\right) \tag{6a}$$

subject to

$$u_i(l|k) \in \Omega_i, \quad k \leq l \leq k+N-1, \tag{6b}$$

$$u_i(l|k) = 0, \quad k+N \leq l, \tag{6c}$$

$$x_i(k) = \widehat{x}_i(k), \tag{6d}$$

$$\forall\, i \in \mathbb{I}_M$$

For open-loop integrating/unstable systems, an additional terminal state constraint that forces the unstable modes to the origin at the end of the control horizon is necessary to ensure stability ([29]).

Definition 1. The *normal cone* to a convex set Ω at a point $x \in \Omega$ is denoted by $N(x, \Omega)$ and defined by

$$N(x; \Omega) = \{s \mid \langle s, y - x \rangle \leq 0 \text{ for all } y \in \Omega\}.$$

Let $(\mathbf{u}_1^*, \mathbf{u}_2^*, \ldots, \mathbf{u}_M^*)$ denote the solution to the centralized optimization problem of Equation (6). By definition, $\mathbf{u}_i^{*\prime} = [\overline{\mathbf{u}}_i^{*\prime}, 0, 0, \ldots]$, $\forall\, i \in \mathbb{I}_M$. For each subsystem i, define $\mathcal{U}_i \in \mathbb{R}^{m_i N}$ as $\mathcal{U}_i = \Omega_i \times \Omega_i \times \ldots \times \Omega_i$. Hence, $\overline{\mathbf{u}}_i^* \in \mathcal{U}_i, \ \forall\, i \in \mathbb{I}_M$. The results presented are valid also for $\Phi(\cdot) = \sum_{i=1}^{M} w_i \Phi_i(\cdot)$ convex and differentiable on some open neighborhood of $\mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_M$ [†]. Optimality is characterized by the following result (which uses convexity but does not necessarily assume that the solution is unique).

Lemma 2. $(\mathbf{u}_1^*, \mathbf{u}_2^*, \ldots, \mathbf{u}_M^*)$ *is optimal for the optimization problem of Equation* (6) *if and only if Equations* (6b) *and* (6c) *hold for each* $i \in \mathbb{I}_M$ *and*

$$-\nabla_{\overline{\mathbf{u}}_i} \Phi\left([\mathbf{u}_1^*, \mathbf{u}_2^*, \ldots, \mathbf{u}_M^*]; \mu(k)\right) \in N(\overline{\mathbf{u}}_i^*; \mathcal{U}_i), \quad \textit{for all } i \in \mathbb{I}_M.$$

[†] The assumptions on $\Phi(\cdot)$ imply that $\Phi([\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M]; \mu(k)) > -\infty$ for all $\mathrm{vec}(u_1(j|k), u_2(j|k), \ldots, u_M(j|k)) \in \Omega_1 \times \Omega_2 \times \ldots \times \Omega_M \ \forall\, j \geq k$ and that $\Phi(\cdot)$ is a proper convex function in the sense of Rockafellar [31, p. 24].

Proof. By definition (Equation (6)), $\mathbf{u}_i' = [\overline{\mathbf{u}}_i', 0, 0, \ldots], \ \forall \ i \in \mathbb{I}_M$. We note that $\Phi(\cdot)$ is a proper convex function, that $\mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_M \subset \text{dom}(\Phi(\cdot))$, and that the relative interior of $\mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_M$ is nonempty (see Rockafellar [31, Theorem 6.2, p. 45]). Hence, the result is a consequence of Rockafellar [31, Theorem 27.4, p. 270]. □

Suppose that the following level set is bounded and closed (hence compact):

$$\mathcal{L} = \Bigg\{ (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M) \Bigg| \Phi([\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M]; \mu(k)) \leq \Phi\left([\mathbf{u}_1^0, \mathbf{u}_2^0, \ldots, \mathbf{u}_M^0]; \mu(k)\right), \\ u_i(k+j|k) \in \Omega_i, \ 0 \leq j \leq N-1, \ u_i(k+j|k) = 0, \ N \leq j, \ i \in \mathbb{I}_M \Bigg\} \tag{7}$$

We have the following result concerning the limiting set of a sequence of normal cones of a closed convex set.

Lemma 3. *Let $\Omega \in \mathbb{R}^n$ be closed and convex. Let $x \in \Omega$ and let $\{x_i\}$ be a sequence of points satisfying $x_i \in \Omega$ and $x_i \to x$. Let $\{v_i\}$ be any sequence satisfying $v_i \in N(x_i; \Omega)$ for all i. Then all limit points of the sequence $\{v_i\}$ belong to $N(x; \Omega)$.*

Proof. Let v be a limit point of $\{v_i\}$ and let $\mathcal{S}$ be a subsequence such that $\lim_{i \in \mathcal{S}} v_i = v$. By definition of normal cone, we have

$$\langle v_i, y - x_i \rangle \leq 0 \ \text{ for all } y \in \Omega \text{ and all } i \in \mathcal{S}.$$

By taking limits as $i \to \infty$, $i \in \mathcal{S}$, we have

$$\langle v, y - x \rangle \leq 0 \ \text{ for all } y \in \Omega,$$

proving that $v \in N(x, \Omega)$, as claimed. □

6 Feasible cooperation-based MPC (FC-MPC): Algorithm and properties

The FC-MPC optimization problem for subsystem $i \in \mathbb{I}_M$ is defined as

$$\mathcal{F}_i \triangleq \min_{\mathbf{u}_i} \quad \sum_{r=1}^{M} w_r \Phi_r \left([\mathbf{u}_1^{p-1}, \ldots, \mathbf{u}_{i-1}^{p-1}, \mathbf{u}_i, \mathbf{u}_{i+1}^{p-1}, \ldots, \mathbf{u}_M^{p-1}]; x_r(k) \right) \tag{8a}$$

subject to

$$u_i(l|k) \in \Omega_i, \ \ k \leq l \leq k+N-1 \tag{8b}$$

$$u_i(l|k) = 0, \ \ k+N \leq l \tag{8c}$$

in which $x_i(k) = \widehat{x}_i(k), \ \forall \ i \in \mathbb{I}_M$. For $\Phi_i(\cdot)$ quadratic and obtained by eliminating the CM states x_i from Equation (5) using the subsystem model (Equation (1)) $\forall \ i \in \mathbb{I}_M$, the

FC-MPC optimization problem (Equation (8)) can be rewritten as

$$\mathcal{F}_i \triangleq \min_{\overline{\mathbf{u}}_i} \quad \frac{1}{2}\overline{\mathbf{u}}_i(k)'\mathfrak{R}_i\overline{\mathbf{u}}_i(k) + \left(r_i(k) + \sum_{j=1,j\neq i}^{M} \mathcal{H}_{ij}\overline{\mathbf{u}}_j^{p-1}(k)\right)' \overline{\mathbf{u}}_i(k) + \text{constant} \tag{9a}$$

subject to

$$u_i(l|k) \in \Omega_i, \quad k \leq l \leq k+N-1 \tag{9b}$$

$$x_i(k) = \widehat{x}_i(k) \tag{9c}$$

in which

$$\mathfrak{R}_i = w_i\mathbb{R}_i + w_iE_{ii}{}'\mathbb{Q}_iE_{ii} + \sum_{j\neq i}^{M} w_jE_{ji}{}'\mathbb{Q}_jE_{ji} \tag{9d}$$

$$\mathcal{H}_{ij} = \sum_{l=1}^{M} w_lE_{li}{}'\mathbb{Q}_lE_{lj} \tag{9e}$$

$$r_i(k) = w_iE_{ii}{}'\mathbb{Q}_if_ix_i(k) + \sum_{j\neq i}^{M} w_jE_{ji}{}'\mathbb{Q}_jf_jx_j(k) \tag{9f}$$

$$E_{ii} = \begin{bmatrix} B_i & 0 & \dots & \dots & 0 \\ A_iB_i & B_i & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_i^{N-1}B_i & \dots & \dots & \dots & B_i \end{bmatrix} \quad E_{ij} = \begin{bmatrix} W_{ij} & 0 & \dots & \dots & 0 \\ A_iW_{ij} & W_{ij} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_i^{N-1}W_{ij} & \dots & \dots & \dots & W_{ij} \end{bmatrix} \quad f_i = \begin{bmatrix} A_i \\ A_i^2 \\ \vdots \\ \vdots \\ A_i^N \end{bmatrix} \tag{9g}$$

$$\mathbb{Q}_i = \text{diag}\left(Q_i(1), \dots, Q_i(N-1), \overline{Q}_i\right) \tag{9h}$$

$$\mathbb{R}_i = \text{diag}\left(R_i(0), R_i(1), \dots, R_i(N-1)\right) \tag{9i}$$

with $\overline{Q}_i$ denoting an appropriately chosen terminal penalty, as described in the sequel.

An implementable algorithm for FC-MPC is now described. Convergence and optimality properties of the proposed FC-MPC algorithm are established subsequently. Some additional notation is required. Let $\Phi([\mathbf{u}_1^p, \mathbf{u}_2^p, \dots, \mathbf{u}_M^p]; \mu(k))$ represent the cooperation-based cost function at iterate p and discrete time k with the initial set of subsystem states $\mu(k)$. Let $\overline{\mathbf{u}}_i^{*(p)}, \forall\ i \in \mathbb{I}_M$ denote the solution to the FC-MPC optimization problem ‡ (Equation (9)). By definition, the corresponding infinite horizon input trajectory $\mathbf{u}_i^{*(p)} =$

‡For notational simplicity, we drop the functional dependence of $\mathbf{u}_i^{*(p)}$ on $\mu(k)$ and $\overline{\mathbf{u}}_j^{p-1}, j \neq i$.

$[\overline{\mathbf{u}}_i^{*(p)\prime}, 0, 0, \ldots], \ \forall\ i \in \mathbb{I}_M$. The cost associated with the input trajectory $\mathbf{u}_i^{*(p)}$ constructed from the solution to $\mathcal{F}_i$ (Equation (9)), is represented as

$$\Phi([\mathbf{u}_1^{p-1}, \ldots, \mathbf{u}_{i-1}^{p-1}, \mathbf{u}_i^{*(p)}, \mathbf{u}_{i+1}^{p-1}, \ldots, \mathbf{u}_M^{p-1}]; \mu(k)).$$

The state sequence for subsystem $i \in \mathbb{I}_M$ generated by the subsystems' input trajectories $(\mathbf{u}_1, \ldots, \mathbf{u}_M)$ and initial set of subsystem states μ is represented as $\mathbf{x}_i\,(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M; \mu)$. For notational convenience, we write $\mathbf{x}_i \leftarrow \mathbf{x}_i\,(\mathbf{u}_1, \ldots, \mathbf{u}_M; \mu)$ and $\overline{\mathbf{x}}_i \leftarrow \overline{\mathbf{x}}_i\,(\overline{\mathbf{u}}_1, \ldots, \overline{\mathbf{u}}_M; \mu)$. At discrete time k, let $p_{\max}(k)$ denote the maximum number of permissible iterates during the sampling interval. By definition, $0 < p(k) \leq p_{\max}(k), \forall\ k \geq 0$.

Algorithm 1. Given $\overline{\mathbf{u}}_i^0, x_i(k), \mathbb{Q}_i, \mathbb{R}_i, \ \forall\ i \in \mathbb{I}_M, p_{\text{max}}(k) \geq 0$ and $\epsilon > 0$
$p \leftarrow 1, e_i \leftarrow \Gamma\epsilon, \Gamma \gg 1$
while $e_i > \epsilon$ for some $i \in \mathbb{I}_M$ and $p \leq p_{\text{max}}(k)$
  do $\forall\ \ i \in \mathbb{I}_M$
    $\overline{\mathbf{u}}_i^{*(p)} \in \ \arg(\text{FC-MPC}_i)$ (see Equation (9))
    $\overline{\mathbf{u}}_i^p \leftarrow w_i \overline{\mathbf{u}}_i^{*(p)} + (1 - w_i)\,\overline{\mathbf{u}}_i^{p-1}$
    Transmit $\overline{\mathbf{u}}_i^p$ to each interconnected subsystem $j \neq i$
    $e_i \leftarrow \|\overline{\mathbf{u}}_i^p - \overline{\mathbf{u}}_i^{p-1}\|$
  end (do)
  $p \leftarrow p + 1$
end (while)

In Algorithm 1 above, the state trajectory for subsystem i at iterate p is obtained as $\overline{\mathbf{x}}_i^p \leftarrow \overline{\mathbf{x}}_i^p\left(\overline{\mathbf{u}}_1^p, \overline{\mathbf{u}}_2^p, \ldots, \overline{\mathbf{u}}_M^p; \mu(k)\right)$. The maximum allowable iterates in each sampling interval $p_{\max}(k)$ is a design limit; one may choose to terminate Algorithm 1 prior to this limit.

Assumptions 2. $p \in \mathbb{I}_+, 0 < p_{\max}(k) \leq p^* < \infty$.

Assumptions 3. $N \geq \max(\alpha, 1)$, in which $\alpha = \max_{i \in \mathbb{I}_M} \alpha_i$ and $\alpha_i \geq 0$ denotes the number of unstable modes for subsystem $i \in \mathbb{I}_M$.

Lemma 4. *Consider the FC-MPC formulation of Equations* (8), (9). *The sequence of cost functions* $\left\{\Phi([\mathbf{u}_1^p, \mathbf{u}_2^p, \ldots, \mathbf{u}_M^p]; \mu(k))\right\}$ *generated by Algorithm 1 is a nonincreasing function of the iteration number p.*

Proof. From Algorithm 1 we know that

$$\Phi\left([\mathbf{u}_1^{p-1}, \ldots \mathbf{u}_{i-1}^{p-1}, \mathbf{u}_i^{*(p)}, \mathbf{u}_{i+1}^{p-1}, \ldots, \mathbf{u}_M^{p-1}]; \mu(k)\right) \leq \Phi\left([\mathbf{u}_1^{p-1}, \mathbf{u}_2^{p-1}, \ldots, \mathbf{u}_M^{p-1}]; \mu(k)\right), \quad \forall\ i \in \mathbb{I}_M$$

Therefore, from the definition of $\mathbf{u}_i^p$ (Algorithm 1) we have

$$\begin{aligned}\Phi\left([\mathbf{u}_1^p,\mathbf{u}_2^p,\ldots,\mathbf{u}_M^p;\mu(k))\right) &= \Phi\Big([w_1\mathbf{u}_1^{*(p)}+(1-w_1)\mathbf{u}_1^{p-1},\ldots,w_M\mathbf{u}_M^{*(p)}+(1-w_M)\mathbf{u}_M^{p-1}];\mu(k)\Big)\\ &= \Phi\Big([w_1(\mathbf{u}_1^{*(p)},\mathbf{u}_2^{p-1},\ldots,\mathbf{u}_M^{p-1})+w_2(\mathbf{u}_1^{p-1},\mathbf{u}_2^{*(p)},\ldots,\mathbf{u}_M^{p-1})+\ldots\\ &\qquad \ldots+w_M(\mathbf{u}_1^{p-1},\mathbf{u}_2^{p-1},\ldots,\mathbf{u}_M^{*(p)})];\mu(k)\Big)\end{aligned}$$

By convexity of $\Phi(\cdot)$,

$$\begin{aligned}&\leq \sum_{r=1}^{M} w_r\Phi\Big([\mathbf{u}_1^{p-1},\ldots,\mathbf{u}_{r-1}^{p-1},\mathbf{u}_r^{*(p)},\mathbf{u}_{r+1}^{p-1},\ldots,\mathbf{u}_M^{p-1}];\mu(k)\Big)\\ &\leq \Phi\Big([\mathbf{u}_1^{p-1},\mathbf{u}_2^{p-1},\ldots,\mathbf{u}_M^{p-1}];\mu(k)\Big)\end{aligned} \tag{10}$$

□

Lemma 5. *All limit points of Algorithm 1 are optimal*

Proof. Let $(\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_M)$ be a limit point and $\mathcal{S}$ be a subsequence for which

$$\lim_{p\in\mathcal{S}}(\mathbf{u}_1^{p-1},\mathbf{u}_2^{p-1},\ldots,\mathbf{u}_M^{p-1})=(\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_M).$$

By definition, $\mathbf{t}_i{}'=[\overline{\mathbf{t}}_i{}',0,0,\ldots],\ \forall\, i\in\mathbb{I}_M$ where $\overline{\mathbf{t}}_i\in\mathcal{U}_i$. By taking a further subsequence of $\mathcal{S}$ if necessary and using compactness of the level set $\mathcal{L}$, we can define $(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_M)$ such that

$$\lim_{p\in\mathcal{S}}(\mathbf{u}_1^{p},\mathbf{u}_2^{p},\ldots,\mathbf{u}_M^{p})=(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_M).$$

We have, $\mathbf{v}_i{}'=[\overline{\mathbf{v}}_i{}',0,0,\ldots],\ \forall\, i\in\mathbb{I}_M$ where $\overline{\mathbf{v}}_i\in\mathcal{U}_i$ From Equation (10) (Lemma 4), by taking limits and noting that $\Phi\left([\mathbf{u}_1^p,\mathbf{u}_2^p,\ldots,\mathbf{u}_M^p]\,;\mu(k)\right)$ is bounded below, we have that

$$\Phi\left([\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_M]\,;\mu(k)\right)=\Phi\left([\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_M]\,;\mu(k)\right)$$

Define $\mathbf{z}_i=\frac{1}{w_i}\mathbf{v}_i-\left(\frac{1}{w_i}-1\right)\mathbf{t}_i,\ \forall\, i\in\ \mathbb{I}_M$, we have by taking limits in Equation (10) (Lemma 4) and using

$$\Phi\Big([\mathbf{u}_1^{p-1},\ldots,\mathbf{u}_{i-1}^{p-1},\mathbf{u}_i^{*(p)},\mathbf{u}_{i+1}^{p-1},\ldots,\mathbf{u}_M^{p-1}];\mu(k)\Big)\leq\Phi\Big([\mathbf{u}_1^{p-1},\mathbf{u}_2^{p-1},\ldots,\mathbf{u}_M^{p-1}];\mu(k)\Big)\quad \forall\, i\in\mathbb{I}_M$$

that in fact

$$\begin{aligned}\Phi\left([\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_M]\,;\mu(k)\right)&=\Phi\left([\mathbf{t}_1,\ldots,\mathbf{t}_{i-1},\mathbf{z}_i,\mathbf{t}_{i+1},\ldots,\mathbf{t}_M]\,;\mu(k)\right)\\&=\Phi\left([\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_M]\,;\mu(k)\right),\ \forall\, i\in\mathbb{I}_M\end{aligned} \tag{11}$$

From Algorithm 1, we have for each i

$$-\nabla_{\overline{\mathbf{u}}_i}\Phi\Big([\mathbf{u}_1^{p-1},\ldots,\mathbf{u}_{i-1}^{p-1},\mathbf{u}_i^{*(p)},\mathbf{u}_{i+1}^{p-1},\ldots,\mathbf{u}_M^{p-1}]\,;\mu(k)\Big)\in N(\overline{\mathbf{u}}_i^{*(p)},\mathcal{U}_i),$$

Taking limits and invoking Lemma 3, we see that

$$-\nabla_{\overline{\mathbf{u}}_i}\Phi\left([\mathbf{t}_1,\ldots,\mathbf{t}_{i-1},\mathbf{z}_i,\mathbf{t}_{i+1},\ldots,\mathbf{t}_M]\,;\mu(k)\right)\in N(\overline{\mathbf{z}}_i,\mathcal{U}_i),\ \forall\, i\in\mathbb{I}_M$$

in which $\mathbf{z}_i{}' = [\overline{\mathbf{z}}_i{}', 0, 0, \ldots]$. From Equation (11), we know

$$\Phi\left([\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_M]\,;\mu(k)\right)=\Phi\left([\mathbf{t}_1,\ldots,\mathbf{t}_{i-1},\mathbf{z}_i,\mathbf{t}_{i+1},\ldots,\mathbf{t}_M]\,;\mu(k)\right)$$

Therefore, $\mathbf{t}_i$ is also a minimizer of $\Phi\left([\mathbf{t}_1,\ldots,\mathbf{t}_{i-1},\mathbf{z}_i,\mathbf{t}_{i+1},\ldots,\mathbf{t}_M]\,;\mu(k)\right)$ over $\mathcal{U}_i$ and so, we have

$$-\nabla_{\overline{\mathbf{u}}_i}\Phi\left([\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_M]\,;\mu(k)\right)\in N(\overline{\mathbf{t}}_i,\mathcal{U}_i),\ \forall\, i\in\mathbb{I}_M$$

Hence, $(\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_M)$ satisfies the optimality condition. □

For the iterates confined to a level set (Equation (7)), a limit point is guaranteed to exist. From Lemma 5, all limit points are optimal. Using strict convexity of the objective in Equation (9), it follows that $(\mathbf{u}_1^*,\ldots,\mathbf{u}_M^*)$ is in fact the limit.

7 Closed-loop properties of FC-MPC under state feedback

At time k, let the FC-MPC scheme be terminated after $p(k)=q$ iterates. Let

$$\mathbf{u}_i^q(\mu(k))' = \left[u_i^q(\mu(k),0)', u_i^q(\mu(k),1)',\ldots,u_i^q(\mu(k),N-1)',\ 0,\ 0,\ldots\right]$$

for each $i\in\mathbb{I}_M$ represent the solution to the FC-MPC algorithm (Algorithm 1) after q iterates. The distributed MPC control law is obtained through a receding horizon implementation of optimal control whereby the input applied to subsystem i is $u_i^q(\mu(k),0)$.

Assumptions 4. For each $i\in\mathbb{I}_M$, (A_{ii},B_{ii}) is stabilizable.

Assumptions 5. $Q_i(0)=Q_i(1)=\ldots=Q_i(N-1)=Q_i>0$ and $R_i(0)=R_i(1)=\cdots=R_i(N-1)=R_i>0,\ \forall\, i\in\mathbb{I}_M$.

Lemmas 4 and 5 lead to the following results on closed-loop stability.

7.1 Stable modes

Feasibility of FC-MPC optimizations and domain of attraction. For open-loop stable systems, the domain of the controller is $\mathbb{R}^n$, $n=\sum_{i=1}^M n_i$. Convexity of each of the admissible input sets $\Omega_i,\ i\in\mathbb{I}_M$ and Algorithm 1 guarantee that if a feasible input trajectory exists for each subsystem $i\in\mathbb{I}_M$ at time $k=0$ and $p(0)=0$, then a feasible input trajectory exists for all subsystems at all future times. One trivial choice for a feasible input trajectory at $k=0$ is $u_i(k+l|k)=0,\ \ l\geq 0,\ \forall\, i\in\mathbb{I}_M$. This choice follows from our assumption that Ω is nonempty and $0\in\text{int}(\Omega)$. The domain of attraction for the closed-loop system is $\mathbb{R}^n$.

Initialization. At time $k = 0$, let $\mathbf{u}_i^0(0) = [0,\ 0, \ldots\ldots]'$, $\forall\ i \in \mathbb{I}_M$. Since $0 \in \text{int}(\Omega)$, this sequence of inputs is feasible. Define $\widetilde{J}_N(\mu(0)) = \Phi([\mathbf{u}_1^0(0), \ldots, \mathbf{u}_M^0(0)]; \mu(0))$ to be the value of the cooperation-based cost function with the set of zero input initialization trajectories and the set of initial subsystem states $\mu(0)$. At time $k > 0$, define $\forall\ i \in \mathbb{I}_M$

$$\mathbf{u}_i^0\,(k)' = \left[u_i^{p(k-1)}(\mu(k-1), k)', \ldots, u_i^{p(k-1)}(\mu(k-1), k+N-2)',\ 0,\ 0, \ldots\right] \tag{12}$$

$(\mathbf{u}_1^0(k), \mathbf{u}_2^0(k), \ldots, \mathbf{u}_M^0(k))$ constitutes a set of feasible subsystem input trajectories with an associated cost function $J_N^0(\mu(k)) = \Phi\left([\mathbf{u}_1^0(k), \mathbf{u}_2^0(k), \ldots, \mathbf{u}_M^0(k)]; \mu(k)\right)$. The value of the cooperation-based cost function after $p(k)$ iterates is denoted by $J_N^{p(k)}(\mu(k)) = \Phi([\mathbf{u}_1^{p(k)}(k), \ldots, \mathbf{u}_M^{p(k)}(k)]; \mu(k))$. The following lemma establishes the relationship between the different cost function values.

Lemma 6. *Given Algorithm 1, employing the FC-MPC optimization problem of Equation (9), for a system with stable decentralized modes. At time $k = 0$, let Algorithm 1 be initialized with input $u_i(k+l|k) = 0,\ l \geq 0,\ \forall\ i \in \mathbb{I}_M$. If for all times $k > 0$, each FC-MPC optimization problem is initialized with the strategy described in Equation (12), then we have,*

$$J_N^{p(k)}(\mu(k)) \leq J_N^0(\mu(k)) \leq \widetilde{J}_N(\mu(0)) - \sum_{j=0}^{k-1}\sum_{i=1}^{M} w_i L_i(x_i(j), 0) \leq \widetilde{J}_N(\mu(0)) \tag{13}$$

$$\forall\ p(k) \geq 0 \text{ and all } k \geq 0$$

A proof is given in Appendix B.

Lemma 6 can be used to show stability in the sense of Lyapunov [39, p. 136]. Attractivity of the origin follows from the cost relationship $0 \leq J_N^{p(k+1)}(\mu(k+1)) \leq J_N^0(\mu(k)) = J_N^{p(k)}(\mu(k)) - \sum_{i=1}^M w_i L_i(x_i(k), u_i^{p(k)}(k))$. Asymptotic stability, therefore, follows.

Assumptions 6. For each $i \in \mathbb{I}_M$, A_{ii} is stable, $\mathbb{Q}_i = \text{diag}\left(Q_i(1), \ldots, Q_i(N-1), \overline{Q}_i\right)$, in which $\overline{Q}_i$ is the solution of the Lyapunov equation $A_i{}'\overline{Q}_i A_i - \overline{Q}_i = -Q_i$

Remark 1. Consider a ball $B_\varepsilon(0), \varepsilon > 0$ such that the input constraints in each FC-MPC optimization problem are inactive. Because $0 \in \text{int}(\Omega_1 \times \cdots \times \Omega_M)$ and the distributed MPC control law is stable and attractive, an $\varepsilon > 0$ exists. Let Assumption 6 hold. For $\mu \in B_\varepsilon(0)$, $\overline{\mathbf{u}}_i^p(\mu), i \in \mathbb{I}_M$ is linear in $x_i, i \in \mathbb{I}_M$. Also, the initialization strategy for Algorithm 1 is independent of μ. The input trajectory $\overline{\mathbf{u}}_i^p(\mu),\ i \in \mathbb{I}_M$ generated by Algorithm 1 is, therefore, a Lipschitz continuous function of μ for all $p \in \mathbb{I}_+$. If $p \leq p^* < \infty$ (Assumption 2), a global Lipschitz constant (independent of p) can be estimated.

A stronger, exponential stability result is established using the following theorem.

Theorem 1 (Stable modes)**.** *Consider Algorithm 1 under state feedback employing the FC-MPC optimization problem of Equation* (9), $\forall\ i \in \mathbb{I}_M$. *Let Assumptions 1 to 6 be satisfied. The origin is an exponentially stable equilibrium for the nominal closed-loop system*

$$x_i(k+1) = A_i x_i(k) + B_i u_i^{p(k)}(\mu(k), 0) + \sum_{j \neq i}^{M} W_{ij} u_j^{p(k)}(\mu(k), 0), \quad i \in \mathbb{I}_M,$$

for all $\mu(0) \in \mathbb{R}^n$ and all $p(k) = 1, 2, \ldots, p_{\max}(k)$.

A proof is given in Appendix B.

7.2 Unstable decentralized modes

From Assumption 1, unstable modes may be present only in the decentralized model. For systems with some decentralized model eigenvalues on or outside the unit circle, closed-loop stability under state feedback can be achieved using a terminal state constraint that forces the unstable decentralized modes to zero at the end of the control horizon. Define

$$\mathbb{S}_i = \{x_{ii} \,|\, \exists\, \overline{\mathbf{u}}_i \in \mathcal{U}_i \text{ such that } {S_{u_i}}'[\mathcal{C}_N(A_{ii}, B_{ii})\, \overline{\mathbf{u}}_i + A_{ii}^N x_{ii}] = 0\} \qquad \text{steerable set}$$

to be the set of decentralized states x_{ii} that can be steered to the origin in N moves. From Assumption 1 and because the domain of each $x_{ij}, i, j \in \mathbb{I}_M, j \neq i$ is $\mathbb{R}^{n_{ij}}$, we define

$$\mathbb{D}_{R_i} = \mathbb{R}^{n_{i1}} \times \cdots \times \mathbb{R}^{n_{i(i-1)}} \times \mathbb{S}_i \times \mathbb{R}^{n_{i(i+1)}} \times \cdots \times \mathbb{R}^{n_{iM}} \subseteq \mathbb{R}^{n_i}, \, i \in \mathbb{I}_M \qquad \text{domain of regulator}$$

to be the of all x_i for which an admissible input trajectory $\overline{\mathbf{u}}_i$ exists that drives the unstable decentralized modes ${U_{u_i}}' x_i$ to zero (in N moves). The domain of the controller for the nominal closed-loop system

$$x_i^+ = A_i x_i + B_i u_i^p(\mu, 0) + \sum_{j \neq i}^{M} u_j^p(\mu, 0), \quad i \in \mathbb{I}_M$$

is given by

$$\mathbb{D}_C = \{\mu \,| x_i \in \mathbb{D}_{R_i}, \; i \in \mathbb{I}_M\} \qquad \text{domain of controller}$$

The set $\mathbb{D}_C$ is positively invariant for the nominal system.

Initialization. Since $\mu(0) \in \mathbb{D}_C$, a feasible input trajectory exists and can be computed by solving the following simple quadratic program (QP) for each $i \in \mathbb{I}_M$.

$$\overline{\mathbf{u}}_i^0 = \arg\min_{\overline{\mathbf{u}}_i} \quad \|\overline{\mathbf{u}}_i\|^2 \tag{14a}$$

subject to

$${U_{u_i}}' \left(\mathcal{C}_N(A_i, B_i)\, \overline{\mathbf{u}}_i + A_i^N x_i(0) \right) = 0 \tag{14b}$$

$$u_i(l|0) \in \Omega_i, \;\; 0 \leq l \leq N-1 \tag{14c}$$

in which U_{u_i} is obtained through a Schur decomposition [12, p. 341] of A_i [§]. Since unstable modes, if any, are present only in the decentralized model (Assumption 1), we have ${U_{u_i}}' \left(\mathcal{C}_N(A_i, B_i)\, \overline{\mathbf{u}}_i + A_i^N x_i(0) \right) = {S_{u_i}}' \left(\mathcal{C}_N(A_{ii}, B_{ii})\, \overline{\mathbf{u}}_i + A_{ii}^N x_{ii}(0) \right)$.

[§] The Schur decomposition of $A_i = \begin{bmatrix} U_{s_i} & U_{u_i} \end{bmatrix} \begin{bmatrix} A_{s_i} & \bigotimes \\ & A_{u_i} \end{bmatrix} \begin{bmatrix} {U_{s_i}}' \\ {U_{u_i}}' \end{bmatrix}$ and $A_{ii} = \begin{bmatrix} S_{s_i} & S_{u_i} \end{bmatrix} \begin{bmatrix} A_{s_{ii}} & \bigoplus \\ & A_{u_{ii}} \end{bmatrix} \begin{bmatrix} {S_{s_i}}' \\ {S_{u_i}}' \end{bmatrix}$. The eigenvalues of $A_{s_i}, A_{s_{ii}}$ are strictly inside the unit circle and the eigenvalues of $A_{u_i}, A_{u_{ii}}$ are on or outside the unit circle.

Feasibility of FC-MPC optimizations and domain of attraction. In the nominal case, the initialization QP (Equation (14)) needs to be solved only once for each subsystem *i.e.*, at time $k = 0$. Nominal feasibility is assured for all $k \geq 0$ and $p(k) > 0$ if the initialization QP at $k = 0$ is feasible for each $i \in \mathbb{I}_M$. At time $k + 1$, the the initial input trajectory is given by Equation (12) for all $i \in \mathbb{I}_M$. The domain of attraction for the closed-loop system is the set $\mathbb{D}_C$.

Consider subsystem i with $\alpha_i \geq 0$ unstable modes. Since all interaction models are stable, all unstable modes arise from the decentralized model matrix A_{ii}. To have a bounded objective, the predicted control trajectory for subsystem i at iterate $p(k)$, $\overline{\mathbf{u}}_i^{p(k)}$, must bring the unstable decentralized modes ${U_{u_i}}' x_i$ to the origin at the end of the control horizon. Boundedness of the infinite horizon objective can be ensured by adding an end constraint ${U_{u_i}}' \left(\mathcal{C}_N(A_i, B_i)\, \overline{\mathbf{u}}_i + A_i^N x_i(k) \right) = 0$ to the FC-MPC optimization problem (Equations (8), (9)) within the framework of Algorithm 1. Feasibility of the above end constraint follows because $N \geq \alpha_i$ and (A_i, B_i) is stabilizable [¶].

At time $k = 0$, let the FC-MPC formulation be initialized with the feasible input trajectory $\mathbf{u}_i^0(0) = [v_i(0)', v_i(1)', \ldots]'$ obtained as the solution to Equation (14), in which $v_i(s) = 0,\ N \leq s,\ \forall\, i \in \mathbb{I}_M$, and let $\overline{J}_N(\mu(0)) = \Phi([\mathbf{u}_1^0(0), \ldots, \mathbf{u}_M^0(0)]; \mu(0))$ denote the associated cost function value. We have $J_N^{p(k)}(\mu(k)) \leq J_N^0(\mu(k)) \leq \overline{J}_N(\mu(0)) - \sum_{j=0}^{k-1} w_i L_i(x_i(k), 0) \leq \overline{J}_N(\mu(0)),\ \forall\ k \geq 0, p(k) > 0$. The proof for the above claim is identical to the proof for Lemma 6. Asymptotic stability can be established using arguments identical to those outlined in Section 7.1.

Assumptions 7. $\alpha > 0$ (see Assumption 3).
For each $i \in \mathbb{I}_M$, $\mathbb{Q}_i = \operatorname{diag}\left(Q_i(1), \ldots, Q_i(N-1), \overline{Q}_i\right)$, in which $\overline{Q}_i = U_{s_i} \Sigma_i {U_{s_i}}'$ with Σ_i obtained as the solution of the Lyapunov equation ${A_{s_i}}' \Sigma_i A_{s_i} - \Sigma_i = -{U_{s_i}}' Q_i U_{s_i}$.

The following theorem establishes exponential stability for systems with unstable decentralized modes.

Theorem 2 (Unstable modes)**.** *Consider Algorithm 1 under state feedback, employing the FC-MPC optimization problem of Equation* (9), $\forall\ i \in \mathbb{I}_M$, *with an additional terminal constraint* ${U_{u_i}}' x_i(k + N|k) = {U_{u_i}}' \left(\mathcal{C}_N(A_i, B_i)\, \overline{\mathbf{u}}_i + A_i^N x_i(k) \right) = 0$ *enforced on the unstable decentralized modes. Let Assumptions 1 to 5 and Assumption 7 hold. The origin is an exponentially stable equilibrium point for the nominal closed-loop system*

$$x_i(k+1) = A_i x_i(k) + B_i u_i^{p(k)}(\mu(k), 0) + \sum_{j \neq i}^{M} W_{ij} u_j^{p(k)}(\mu(k), 0), \quad i \in \mathbb{I}_M,$$

for all $\mu(0) \in \mathbb{D}_C$ *and all* $p(k) = 1, 2, \ldots, p_{\max}(k)$.

A proof is provided in Appendix B.
For positive semidefinite penalties on the states $x_i, i \in \mathbb{I}_M$, we have the following results:

Remark 2. Let Assumption 1 hold and let $Q_i \geq 0,\ (A_i, Q_i^{1/2}),\ i \in \mathbb{I}_M$ be detectable. The nominal closed-loop system $x_i(k+1) = A_i x_i(k) + B_i u_i^{p(k)}(\mu(k), 0) + \sum_{j \neq i} W_{ij} u_j^{p(k)}(\mu(k), 0),\ i \in$

[¶] Stabilizability of (A_i, B_i) follows from Assumptions 1 and 4.

$\mathbb{I}_M$ is exponentially stable under the state feedback distributed MPC control law defined by either Theorem 1 or Theorem 2.

Remark 3. Let Assumption 1 hold and let $Q_i = \text{diag}(Q_{1i}, \ldots, Q_{Mi})$ with $Q_{ii} > 0, Q_{ij} \geq 0,\ \forall i, j \in \mathbb{I}_M, j \neq i$. The nominal closed-loop system $x_i(k+1) = A_i x_i(k) + B_i u_i^{p(k)}(\mu(k), 0) + \sum_{j \neq i} W_{ij} u_j^{p(k)}(\mu(k), 0),\ i \in \mathbb{I}_M$ is exponentially stable under the state feedback distributed MPC control law defined by either Theorem 1 or Theorem 2.

8 Examples

Controller performance index. For the examples presented in this paper, the controller performance index for each systemwide control configuration is calculated as

$$\Lambda_{\text{cost}}(k) = \frac{1}{k} \sum_{j=0}^{k} \sum_{i=1}^{M} \frac{1}{2} \left[x_i(j)' Q_i x_i(j) + u_i(j)' R_i u_i(j) \right]$$

$$\text{in which } Q_i = {C_i}' Q_{y_i} C_i + \varepsilon_i I \geq 0$$

$$R_i > 0, \qquad \varepsilon_i \geq 0$$

$$\Delta\Lambda_{\text{cost}}(\text{config})\% = \frac{\Lambda_{\text{cost}}(\text{config}) - \Lambda_{\text{cost}}(\text{cent})}{\Lambda_{\text{cost}}(\text{cent})} \times 100$$

and k is the simulation time.

8.1 Distillation column control

Table 1: Distillation column model. Bound constraints on inputs L and V. Regulator parameters for MPCs.

$G_{11} = \dfrac{32.63}{(99.6s+1)(0.35s+1)}$	$G_{12} = \dfrac{-33.89}{(98.02s+1)(0.42s+1)}$
$G_{21} = \dfrac{34.84}{(110.5s+1)(0.03s+1)}$	$G_{12} = \dfrac{-18.85}{(75.43s+1)(0.3s+1)}$

$$\begin{bmatrix} T_{21} \\ T_7 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{32} & G_{22} \end{bmatrix} \begin{bmatrix} V \\ L \end{bmatrix} \qquad \begin{matrix} -1.5 \leq V \leq 1.5 \\ -2 \leq L \leq 2 \end{matrix}$$

$Q_{y_1} = 50$	$Q_{y_2} = 50$
$R_1 = 1$	$R_2 = 1$
$\varepsilon_1 = 10^{-6}$	$\varepsilon_2 = 10^{-6}$

Consider the distillation column of [27, p. 813]. Tray temperatures act as inferential variables for composition control. The outputs T_{21}, T_7 are the temperatures of trays 21 and 7 respectively and the inputs L, V denote the reflux flowrate and the vapor boilup flowrate to the distillation column. The implications of the relative gain array (RGA) elements on controller design has been studied in [35]. While the RGA for this system

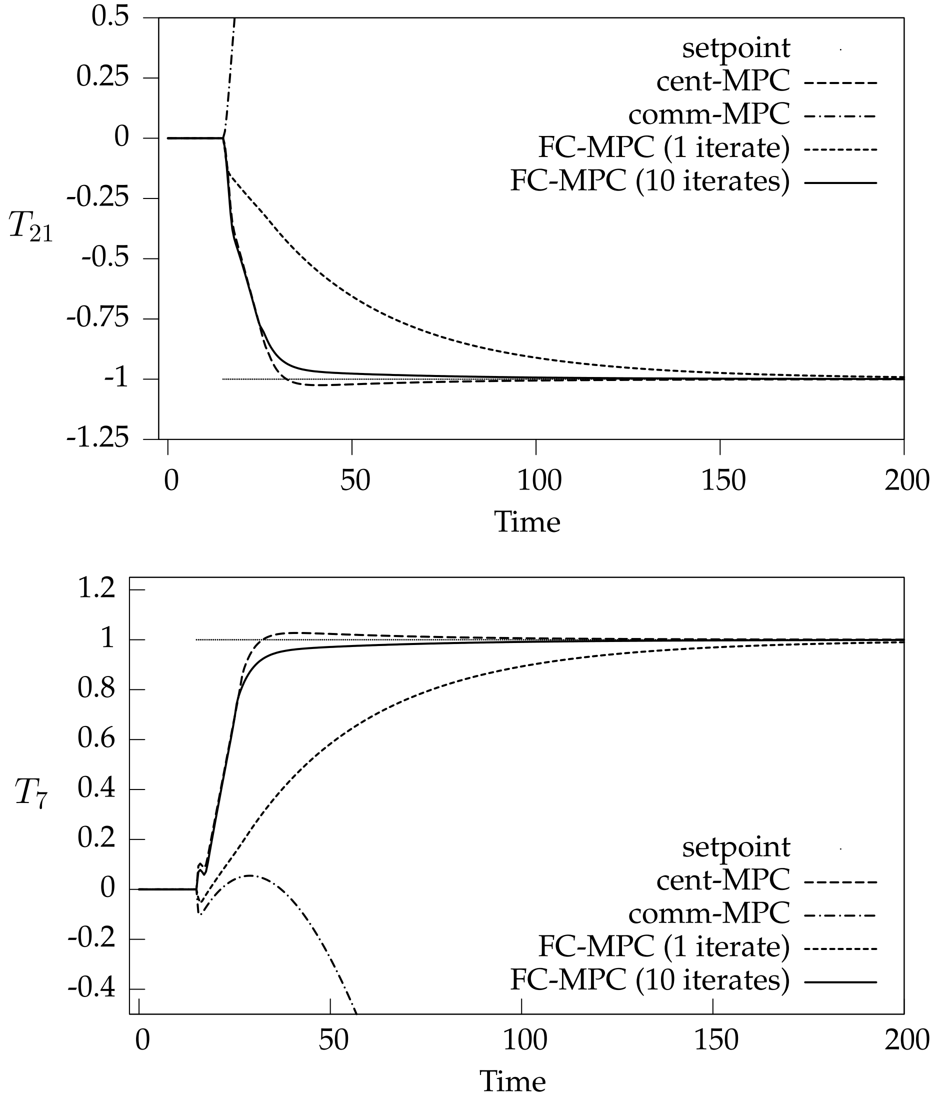

Figure 4: Setpoint tracking performance of centralized MPC, communication-based MPC and FC-MPC. Tray temperatures of the distillation column ([27]).

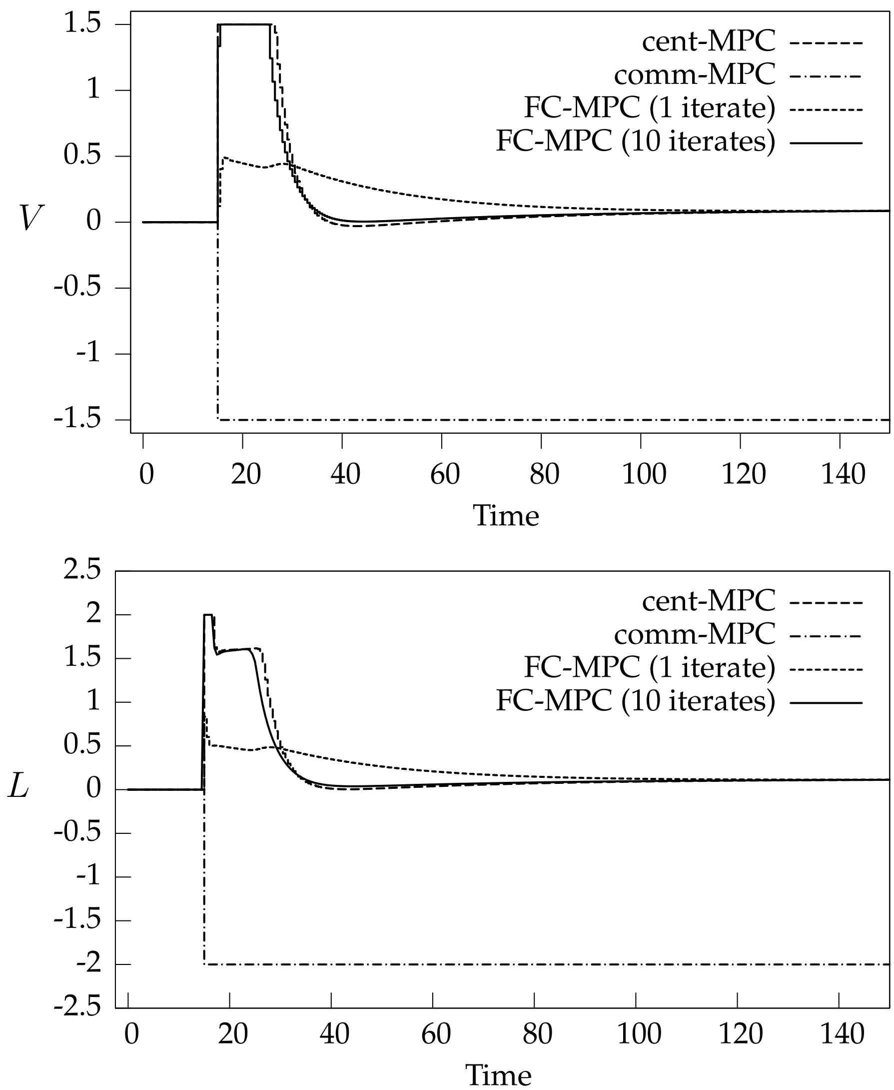

Figure 5: Setpoint tracking performance of centralized MPC, communication-based MPC and FC-MPC. Input profile (V and L) for the distillation column ([27]).

suggests pairing L with T_{21} and V with T_7, we intentionally choose a bad control variable–manipulated variable pairing. While dealing with subsystem-based control of large-scale systems, situations arise in which an optimal pairing policy for the controlled and manipulated variable sets (CVs and MVs) either does not exist or is infeasible due to physical or operational constraints. Such situations are not uncommon. From a control perspective, one prerequisite of a reliable subsystem-based systemwide control strategy is the ability to overcome bad CV-MV choices. Figures 4 and 5 depict the closed-loop performance of centralized MPC (cent-MPC), communication-based MPC (comm-MPC) and FC-MPC when the temperature setpoint of trays 21 and 7 are altered by $-1°C$ and $1°C$ respectively. For each MPC, a control horizon $N = 25$ is used. The nominal plant model, regulator parameters and constraints are given in Table 1.

Table 2: Closed-loop performance comparison of centralized MPC, decentralized MPC, communication-based MPC and FC-MPC.

	Λ_{cost}	$\Delta\Lambda_{\text{cost}}\%$
Cent-MPC	1.72	0
Comm-MPC	∞	∞
FC-MPC (1 iterate)	6.35	269.2
FC-MPC (10 iterates)	1.74	1.32

In the comm-MPC framework, inputs V and L saturate at their constraints and the resulting controller is closed-loop unstable. The situation here is similar to that depicted in Figure 3 (Example 3). The distributed controller derived by terminating the FC-MPC algorithm after one iterate stabilizes the closed-loop system. The closed-loop performance of this distributed controller is significantly worse than the performance of centralized MPC however. The control costs using different MPCs are given in Table 2. FC-MPC after 10 iterates achieves performance that is within $\sim 1.4\%$ of centralized MPC performance.

8.2 Two reactor chain with flash separator

We consider a plant consisting of two continuous stirred tank reactors (CSTRs) followed by a nonadiabatic flash. A schematic of the plant is shown in Figure 6. In each of the CSTRs, the desired product B is produced through the irreversible first order reaction $A \xrightarrow{k_1} B$. An undesirable side reaction $B \xrightarrow{k_2} C$ results in the consumption of B and in the production of the unwanted side product C. The product stream from CSTR-2 is sent to a nonadiabatic flash to separate the excess A from the product B and the side product C. Reactant A has the highest relative volatility and is the predominant component in the vapor phase. A fraction of the vapor phase is purged and the remaining (A rich) stream is condensed and recycled back to CSTR-1. The liquid phase (exiting from the flash) consists mainly of B and C. The first principles model and parameters for the plant are given in Tables 3 to 5. A linear model for the plant is obtained by linearizing the plant around the steady state corresponding to the maximum yield of B, which is the desired operational objective.

Table 3: First principles model for the plant consisting of two CSTRs and a nonadiabatic flash.

Reactor-1:

$$\frac{dH_r}{dt} = \frac{1}{\rho A_r}\left[F_0 + D - F_r\right] \tag{15a}$$

$$\frac{dx_{A_r}}{dt} = \frac{1}{\rho A_r H_r}\left[F_0(x_{A_0} - x_{A_r}) + D(x_{A_d} - x_{A_r})\right] - k_{1_r} x_{A_r} \tag{15b}$$

$$\frac{dx_{B_r}}{dt} = \frac{1}{\rho A_r H_r}\left[F_0(x_{B_0} - x_{B_r}) + D(x_{B_d} - x_{B_r})\right] + k_{1_r} x_{A_r} - k_{2_r} x_{B_r} \tag{15c}$$

$$\frac{dT_r}{dt} = \frac{1}{\rho A_r H_r}\left[F_0(T_0 - T_r) + D(T_d - T_r)\right] - \frac{1}{C_p}\left[k_{1_r} x_{A_r}\Delta H_1 + k_{2_r} x_{B_r}\Delta H_2\right] + \frac{Q_r}{\rho A_r C_p H_r} \tag{15d}$$

Reactor-2:

$$\frac{dH_m}{dt} = \frac{1}{\rho A_m}\left[F_r + F_1 - F_m\right] \tag{15e}$$

$$\frac{dx_{A_m}}{dt} = \frac{1}{\rho A_m H_m}\left[F_r(x_{A_r} - x_{A_m}) + F_1(x_{A_1} - x_{A_m})\right] - k_{1_m} x_{A_m} \tag{15f}$$

$$\frac{dx_{B_m}}{dt} = \frac{1}{\rho A_m H_m}\left[F_r(x_{B_r} - x_{B_m}) + F_1(x_{B_1} - x_{B_m})\right] + k_{1_m} x_{A_m} - k_{2_m} x_{B_m} \tag{15g}$$

$$\frac{dT_m}{dt} = \frac{1}{\rho A_m H_m}\left[F_r(T_r - T_m) + F_1(T_0 - T_m)\right] - \frac{1}{C_p}\left[k_{1_m} x_{A_m}\Delta H_1 + k_{2_m} x_{B_m}\Delta H_2\right] + \frac{Q_m}{\rho A_m C_p H_m} \tag{15h}$$

Nonadiabatic flash:

$$\frac{dH_b}{dt} = \frac{1}{\rho_b A_b}\left[F_m - F_b - D - F_p\right] \tag{15i}$$

$$\frac{dx_{A_b}}{dt} = \frac{1}{\rho_b A_b H_b}\left[F_m(x_{A_m} - x_{A_b}) - (D + F_p)(x_{A_d} - x_{A_b})\right] \tag{15j}$$

$$\frac{dx_{B_b}}{dt} = \frac{1}{\rho_b A_b H_b}\left[F_m(x_{B_m} - x_{B_b}) - (D + F_p)(x_{B_d} - x_{B_b})\right] \tag{15k}$$

$$\frac{dT_b}{dt} = \frac{1}{\rho_b A_b H_b}\left[F_m(T_m - T_b)\right] + \frac{Q_b}{\rho_b A_b H_b C_{p_b}} \tag{15l}$$

$$F_r = k_r\sqrt{H_r} \qquad F_m = k_m\sqrt{H_m} \qquad k_{1_r} = k_1^* \exp\left(\frac{-E_1}{RT_r}\right)$$

$$F_b = k_b\sqrt{H_b} \qquad x_{C_r} = 1 - x_{A_r} - x_{B_r} \qquad k_{2_r} = k_2^* \exp\left(\frac{-E_2}{RT_r}\right)$$

$$x_{C_m} = 1 - x_{A_m} - x_{B_m} \qquad x_{C_b} = 1 - x_{A_b} - x_{B_b} \qquad k_{1_m} = k_1^* \exp\left(\frac{-E_1}{RT_m}\right)$$

$$x_{A_d} = \frac{\alpha_A x_{A_b}}{\Sigma} \qquad x_{B_d} = \frac{\alpha_B x_{B_b}}{\Sigma} \qquad k_{2_r} = k_2^* \exp\left(\frac{-E_2}{RT_m}\right)$$

$$x_{C_d} = \frac{\alpha_C x_{C_b}}{\Sigma} \qquad \Sigma = \alpha_A x_{A_b} + \alpha_B x_{B_b} + \alpha_C x_{C_b}$$

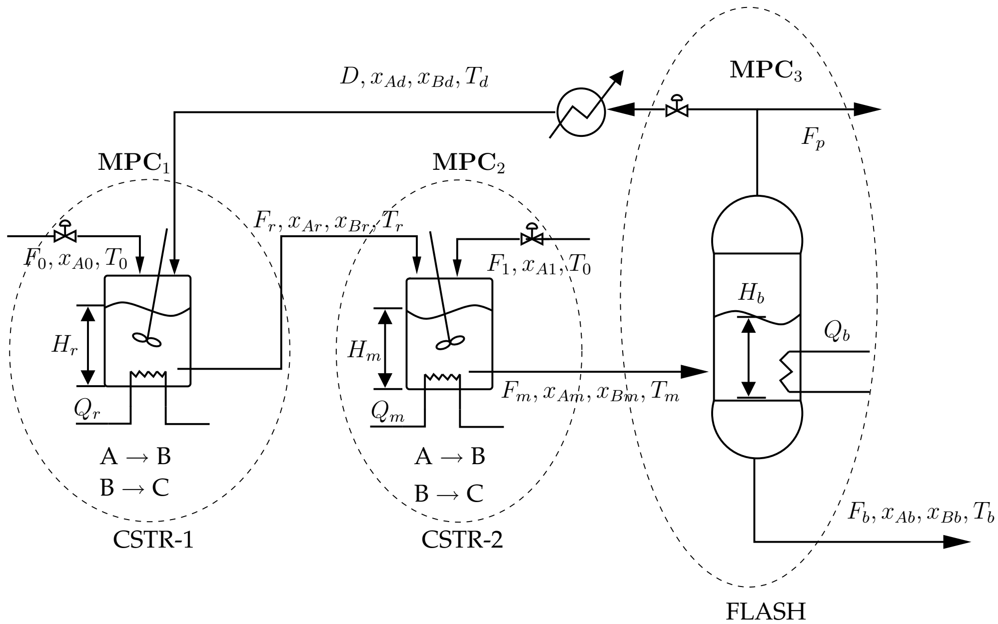

Figure 6: Two reactor chain followed by nonadiabatic flash. Vapor phase exiting the flash is predominantly A. Exit flows are a function of the level in the reactor/flash.

For distributed MPC, 3 MPCs, one each for the two CSTRs and one for the nonadiabatic flash, are used. Under centralized MPC, a single MPC controls the entire plant. The manipulated variables (MVs) for CSTR-1 are the feed flowrate F_0 and the cooling duty Q_{r}. The measured variables are the level of liquid in the reactor H_{r}, the exit mass fractions of A and B *i.e.*, $x_{\mathrm{A_r}}, x_{\mathrm{B_r}}$ respectively and the reactor temperature T_{r}. The controlled variables (CVs) for CSTR-1 are H_{r} and T_{r}. The MVs for CSTR-2 are the feed flowrate F_1 and the reactor cooling load Q_{m}. The (local) measured variables are the level H_{m}, the mass fractions of A and B $x_{\mathrm{A_m}}, x_{\mathrm{B_m}}$ at the outlet, and the reactor temperature T_{m} . The CVs are H_{m} and T_{m}. For the nonadiabatic flash, the MVs are the recycle flowrate D and the cooling duty for the flash Q_{b}. The CVs are the level in the flash H_{b} and the temperature T_{b}. The measurements are $H_{\mathrm{b}}, T_{\mathrm{b}}$ and the product stream mass fractions of A and B ($x_{\mathrm{A_b}}$ and $x_{\mathrm{B_b}}$).

The performance of centralized MPC (Cent-MPC), communication-based MPC (Comm-MPC) and FC-MPC are evaluated when a setpoint change corresponding to a 42% increase in the level H_m is made at time 15. The control horizon for each MPC is $N = 15$. Figures 7 and 8 depict the performance of the different MPC frameworks for the prescribed setpoint change. In the Comm-MPC framework, the flowrate F_1 switches continually between its upper and lower bounds. Subsequently, Comm-MPC leads to unstable closed-loop performance. Both Cent-MPC and FC-MPC (1 iterate) stabilize the closed-loop system. In response to an increase in the setpoint of H_m, the FC-MPC for CSTR-2 orders a maximal increase in flowrate F_1. The flowrate F_1, therefore, saturates at its upper limit. The FC-MPCs for CSTR-1 and the flash cooperate with the FC-MPC for CSTR-2 by initially in-

Table 4: Steady-state parameters for Example 8.2. The operational steady state corresponds to maximum yield of B.

$\rho = \rho_b = 0.15$ Kg m^{-3}	$\alpha_A = 3.5$	$\alpha_B = 1.1$
$\alpha_C = 0.5$	$k_1^* = 0.02$ sec^{-1}	$k_2^* = 0.018$ sec^{-1}
$A_r = 0.3$ m^2	$A_m = 3$ m^2	$A_b = 5$ m^2
$F_0 = 2.667$ Kg sec^{-1}	$F_1 = 1.067$ Kg sec^{-1}	$D = 30.74$ Kg sec^{-1}
$F_p = 0.01D$	$T_0 = 313$ K	$T_d = 313$ K
$C_p = C_{p_b} = 25$ KJ (Kg K)$^{-1}$	$Q_{\mathrm{r}} = Q_{\mathrm{m}} = Q_{\mathrm{b}} = -2.5$ KJ sec^{-1}	$x_{A_0} = 1$
$x_{B_0} = x_{C_0} = 0$	$x_{A_1} = 1$	$x_{B_1} = x_{C_1} = 0$
$\Delta H_1 = -40$ KJ Kg^{-1}	$\Delta H_2 = -50$ KJ Kg^{-1}	$\frac{E_1}{R} = \frac{E_2}{R} = 150$K
$k_r = 2.5$ Kg sec^{-1}m$^{-\frac{1}{2}}$	$k_m = 2.5$ Kg sec^{-1}m$^{-\frac{1}{2}}$	$k_b = 1.5$ Kg sec^{-1}m$^{-\frac{1}{2}}$

Table 5: Input constraints for Example 8.2. The symbol Δ represents a deviation from the corresponding steady-state value.

$-0.1 \leq \Delta F_0 \leq 0.1$	$-0.15 \leq \Delta Q_r \leq 0.15$
$-0.04 \leq \Delta F_1 \leq 0.04$	$-0.15 \leq \Delta Q_r \leq 0.15$
$-0.1 \leq \Delta D \leq 0.1$	$-0.15 \leq \Delta Q_b \leq 0.15$

Table 6: Closed-loop performance comparison of centralized MPC, communication-based MPC and FC-MPC.

	$\Lambda_{\mathrm{cost}} \times 10^{-2}$	$\Delta\Lambda_{\mathrm{cost}}\%$
Cent-MPC	2.0	0
Comm-MPC	∞	∞
FC-MPC (1 iterate)	2.13	6
FC-MPC (10 iterates)	2.01	0.09

creasing F_0 and later increasing D respectively. This feature *i.e.*, cooperation among MPCs is absent under Comm-MPC and is the likely reason for its failure. A performance comparison of the different MPC frameworks is given in Table 6. If Algorithm 1 is terminated after just 1 iterate, FC-MPC incurs a performance loss of 6% compared to cent-MPC. If 10 iterates per sampling interval are possible, the performance of FC-MPC is almost identical to cent-MPC.

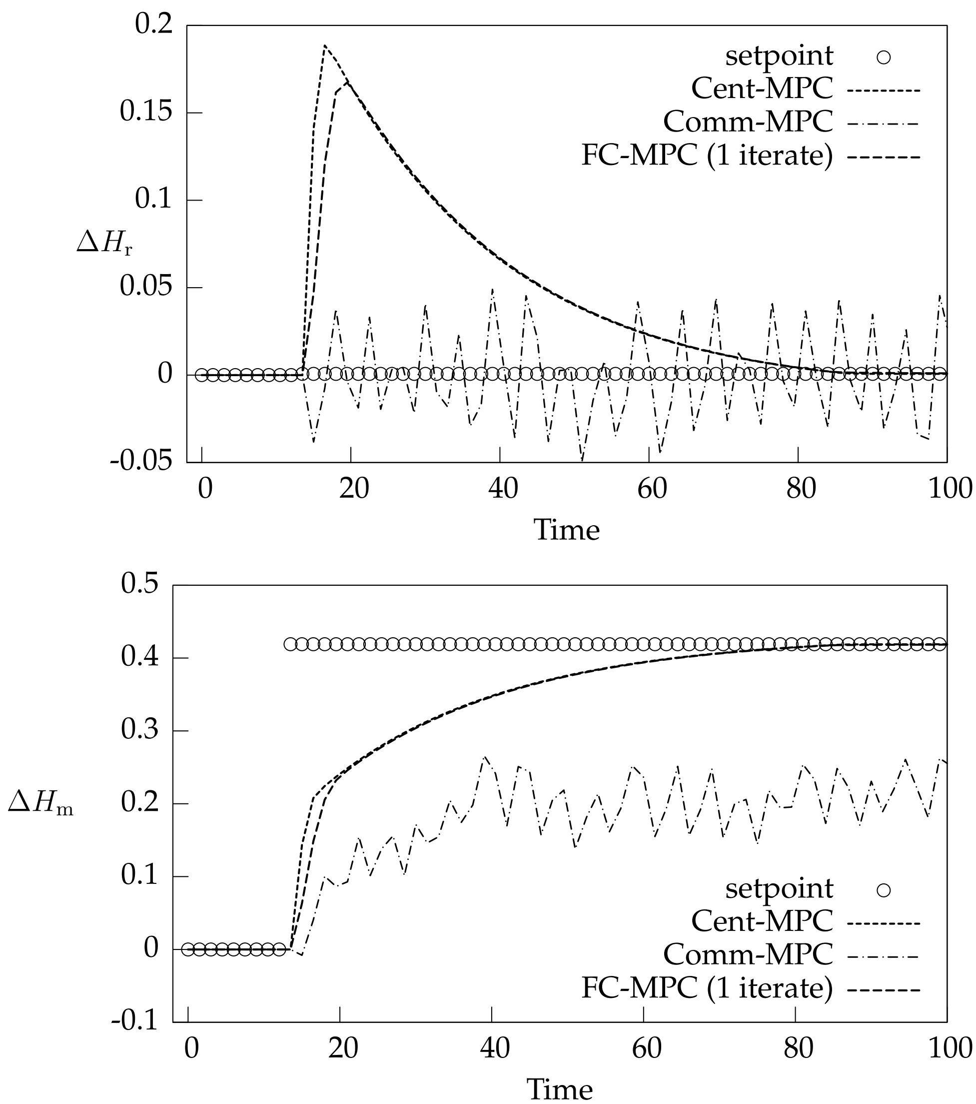

Figure 7: Performance of cent-MPC, comm-MPC and FC-MPC when the level setpoint for CSTR-2 is increased by 42%. Setpoint tracking performance of levels H_r and H_m.

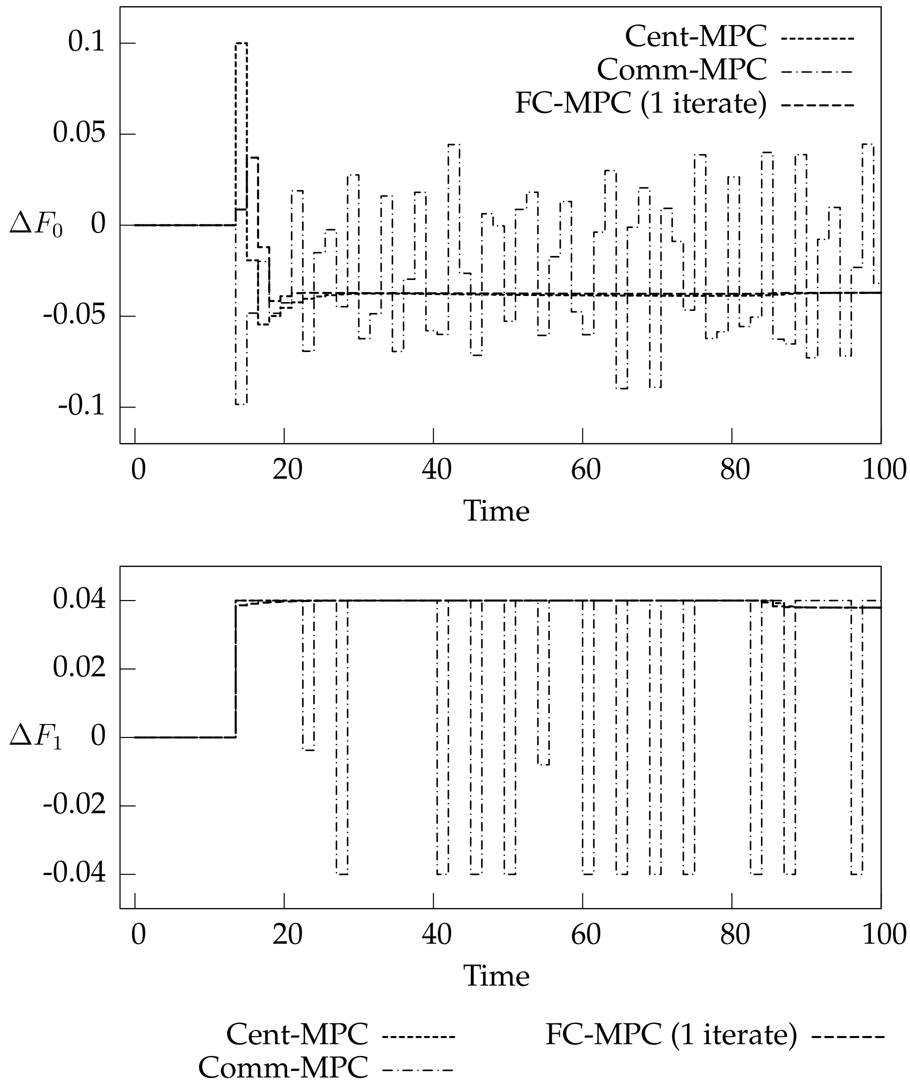

Figure 8: Performance of cent-MPC, comm-MPC and FC-MPC when the level setpoint for CSTR-2 is increased by 42%. Setpoint tracking performance of input flowrates F_0 and F_m.

8.3 Unstable three subsystem network

Table 7: Nominal plant model for Example 5 (Section 8.3). Three subsystems, each with an unstable decentralized pole. Input constraints and regulator parameters.

$$G_{11} = \begin{bmatrix} \frac{s-0.75}{(s+10)(s-0.01)} & \frac{0.5}{(s+11)(s+2.5)} \\ \frac{0.32}{(s+6.5)(s+5.85)} & \frac{1}{(s+3.75)(s+4.5)} \end{bmatrix} \qquad G_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$G_{13} = \begin{bmatrix} \frac{s-5.5}{(s+2.5)(s+3.2)} \\ \frac{0.3}{(s+11)(s+27)} \end{bmatrix} \qquad G_{21} = \begin{bmatrix} \frac{s-0.3}{(s+6.9)(s+3.1)} & \frac{0.31}{(s+41)(s+34)} \\ \frac{-0.19}{(s+16)(s+5)} & \frac{0.67(s-1)}{(s+12)(s+7)} \end{bmatrix}$$

$$G_{22} = \begin{bmatrix} \frac{s-0.5}{(s+20)(s+25)} & \frac{0.6}{(s+14)(s+15)} \\ \frac{-0.33}{(s+3.0)(s+3.1)} & \frac{s-1.5}{(s+20.2)(s-0.05)} \end{bmatrix} \qquad G_{23} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$G_{31} = \begin{bmatrix} 0 & 0 \end{bmatrix} \qquad G_{32} = \begin{bmatrix} \frac{0.9}{(s+17)(s+10.8)} & \frac{-0.45}{(s+26)(s+5.75)} \end{bmatrix}$$

$$G_{33} = \begin{bmatrix} \frac{s-3}{(s+12)(s-0.01)} \end{bmatrix} \qquad y_{\mathrm{I}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y_{\mathrm{II}} = \begin{bmatrix} y_3 \\ y_4 \end{bmatrix} \qquad y_{\mathrm{III}} = \begin{bmatrix} y_5 \end{bmatrix}$$

$$u_{\mathrm{I}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad u_{\mathrm{II}} = \begin{bmatrix} u_3 \\ u_4 \end{bmatrix}$$

$$u_{\mathrm{III}} = \begin{bmatrix} u_5 \end{bmatrix} \qquad \begin{bmatrix} y_{\mathrm{I}} \\ y_{\mathrm{II}} \\ y_{\mathrm{III}} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} u_{\mathrm{I}} \\ u_{\mathrm{II}} \\ u_{\mathrm{III}} \end{bmatrix}$$

$$-1 \leq u_1 \leq 1$$
$$-0.15 \leq u_2 \leq 0.15$$
$$-1.5 \leq u_3 \leq 1.5$$
$$-0.2 \leq u_4 \leq 0.2$$
$$-0.75 \leq u_5 \leq 0.75$$

$Q_{y_1} = 25$	$Q_{y_2} = 25$	$Q_{y_3} = 1$
$R_1 = 1$	$R_2 = 1$	$R_3 = 1$
$\varepsilon_1 = 10^{-6}$	$\varepsilon_2 = 10^{-6}$	$\varepsilon_3 = 10^{-6}$

Consider a plant consisting of three subsystems. The nominal subsystem models, input constraints and controller parameters are given in Table 7. For each MPC, a control horizon $N = 15$ is used. Since each of the subsystems has an unstable decentralized mode, a terminal state constraint that forces the unstable mode to the origin at the end of the control horizon is employed (Theorem 2 for the FC-MPC framework). A setpoint change of 1 and -1 is made to outputs y_1 and y_5 respectively at time = 6. The performance of the distributed controller derived by terminating the FC-MPC algorithm after 1 and 5 iterates respectively is shown in Figures 9-11. A closed-loop performance comparison of the different MPCs for the described setpoint change is given in Table 8. The performance loss under FC-MPC terminated after just 1 iterate is $\sim 14\%$, which is a substantial im-

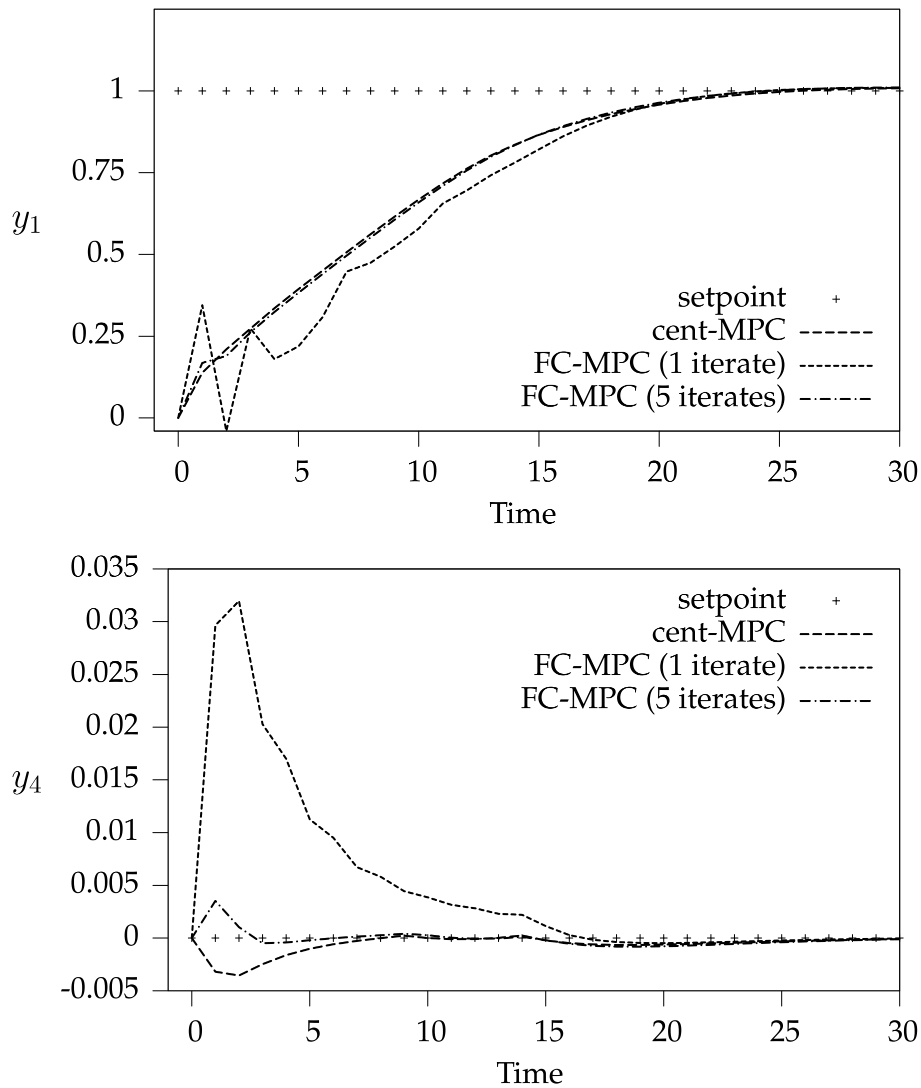

Figure 9: Performance of centralized MPC and FC-MPC for the setpoint change described in Example 8.3. Setpoint tracking performance of outputs y_1 and y_4.

Table 8: Closed-loop performance comparison of centralized MPC, decentralized MPC, communication-based MPC and FC-MPC.

	Λ_{cost}	$\Delta\Lambda_{\text{cost}}\%$
Cent-MPC	1.78	0
Decent-MPC	3.53	98.3
Comm-MPC	3.53	98.2
FC-MPC (1 iterate)	2.03	13.9
FC-MPC (5 iterates)	1.8	0.8

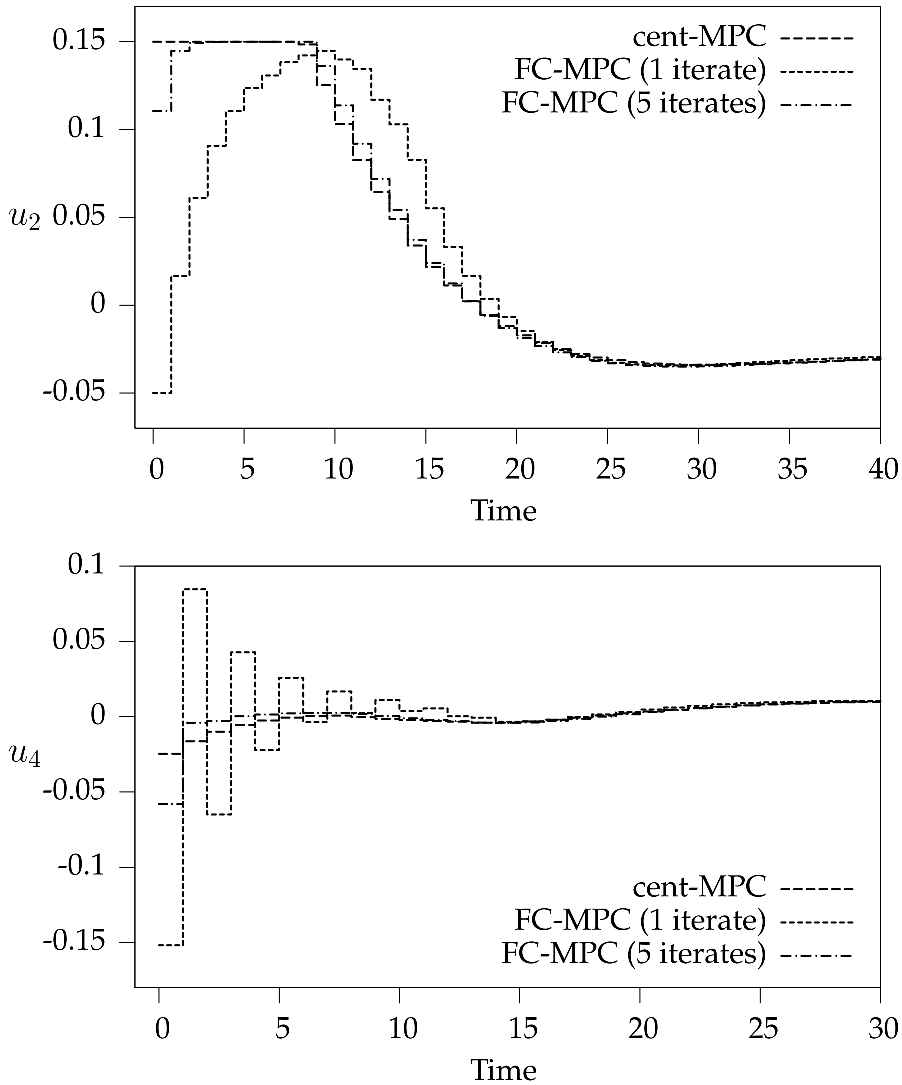

Figure 10: Performance of centralized MPC and FC-MPC for the setpoint change described in Example 8.3. Inputs u_2 and u_4.

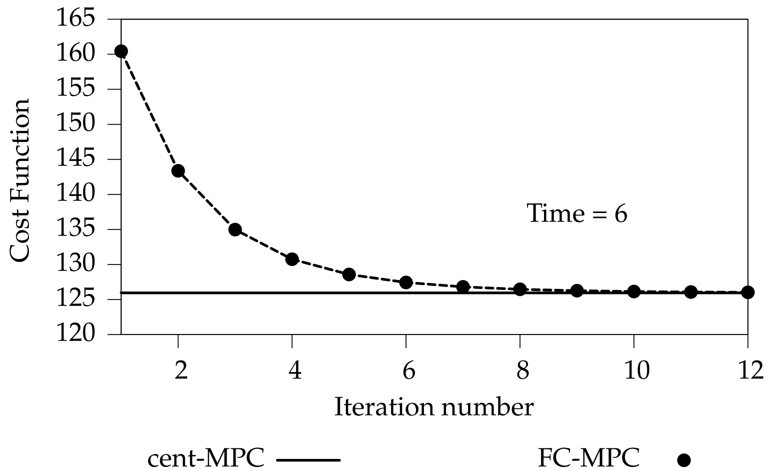

Figure 11: Behavior of the FC-MPC cost function with iteration number at time 6. Convergence to the optimal, centralized cost is achieved after ~ 10 iterates.

provement over decentralized and communication-based MPC. Both decentralized and communication-based MPC incur a performance loss of $\sim 98\%$ relative to centralized MPC. The behavior of the cooperation-based cost function with iteration number at time = 6 is shown in Figure 11. At time = 6, convergence to the centralized MPC solution is achieved after ~ 10 iterates.

9 Discussion and conclusions

A new distributed, linear MPC framework with guaranteed feasibility, optimality and closed-loop stability properties was presented. It is shown that communication-based MPC strategies are unreliable for systemwide control and can lead to closed-loop instability. A cooperation-based distributed MPC algorithm was proposed. The intermediate iterates generated by this cooperation-based MPC algorithm are feasible and the state feedback distributed MPC control law based on any intermediate iterate is nominally closed-loop stable. Therefore, one can terminate Algorithm 1 at end of each sampling interval, irrespective of convergence. At each time k, the states of each subsystem are relayed to all the interconnected subsystems' MPCs. At each iterate p, the MPC for subsystem $i \in \mathbb{I}_M$ calculates its optimal input trajectory $\mathbf{u}_i^p$ assuming the input trajectories generated by the interacting subsystems' MPCs remain at $\mathbf{u}_j^{p-1}$, $\forall\, j \neq i$. The recomputed trajectory $\mathbf{u}_i^p$ is subsequently communicated to each interconnected subsystem's MPC.

Implementation. For a plant with M subsystems employing decentralized MPCs, conversion to the FC-MPC framework involves the following tasks. First, the interaction models must be identified. Techniques for identifying the interaction models under closed-loop operating conditions have been described in [13]. Next, the Hessian and the linear term in the QP for each subsystem MPC need to be modified as shown in Equation 8. Notice that in the decentralized MPC framework, the linear term in the QP is modified after each time step; in the FC-MPC framework, the linear term is updated after each iterate. The Hessian in both frameworks is a constant and requires modification only if the models change. Also, unlike centralized MPC, both decentralized MPC and FC-MPC do not require any information regarding the constraints on the external input variables. Finally, a communication protocol must be established for relaying subsystem state information (after each time step) and input trajectory information (after each iterate). This communication protocol can range from data transfer over wireless networks to storing and retrieval of information from a central database. One may also consider utilizing recent developments in technology for control over networks [3; 5; 17] to establish a communication protocol suitable for distributed MPC. This issue is beyond the scope of this work and remains an open research area. Along similar lines, developing reliable buffer strategies in the event of communication disruptions is another important research problem.

The FC-MPC framework allows the practitioner to seamlessly transition from completely decentralized control to completely centralized control. For each subsystem i, by setting $w_i = 1$, $w_j = 0$, $j \neq i$, and by switching off the communication between the subsystems' MPCs, the system reverts to decentralized MPC. On the other hand, iterating Algorithm 1 to convergence gives the optimal, centralized MPC solution. By terminating Algorithm 1 at intermediate iterates, we obtain performance that lies between the decentralized MPC (base case) and centralized MPC (best case) performance limits allowing the practitioner to investigate the potential control benefits of centralized control without requiring the large control system restructuring and maintenance effort needed to implement and maintain centralized MPC. Taking subsystems off line and bringing subsystems back online are accomplished easily in the FC-MPC framework. Through simple modifications, the FC-MPC framework can be geared to focus on operational objectives (at the expense of optimality), in the spirit of modular multivariable control [24]. For instance, it is possible to modify the FC-MPC framework such that only local inputs are used to track a certain output variable. Details of such a modified FC-MPC framework are available in [38].

Extensions. The FC-MPC formulation can be employed for control of systems with coupled subsystem input constraints of the form $\sum_{i=1}^{M} H_i u_i \leq h, h > 0$. At time k and iterate

p, the FC-MPC optimization problem for subsystem $i \in \mathbb{I}_M$ is

$$\mathbf{u}_i^{*(p)}(k) \in \arg \min_{\mathbf{u}_i} \sum_{r=1}^{M} w_r \Phi_r \left(\mathbf{u}_1^{p-1}, \ldots, \mathbf{u}_{i-1}^{p-1}, \mathbf{u}_i, \mathbf{u}_{i+1}^{p-1}, \ldots, \mathbf{u}_M^{p-1}; x_r(k) \right) \tag{16a}$$

subject to

$$u_i(l|k) \in \Omega_i, \;\; k \leq l \leq k+N-1 \tag{16b}$$

$$u_i(l|k) = 0, \;\; k+N \leq l \tag{16c}$$

$$H_i u_i(l|k) + \sum_{j=1, j\neq i}^{M} H_j u_j^{p-1}(l|k) \leq h, \;\; k \leq l \leq k+N-1 \tag{16d}$$

It can be shown that the sequence of cost functions generated by Algorithm 1 (solving the optimization problem of Equation (16) instead) is a nonincreasing function of the iteration number and converges. Also, the distributed MPC control law based on any intermediate iterate is guaranteed to be feasible and closed-loop stable. Let Φ^∞ be the converged cost function value and let $S^\infty = \{(\mathbf{u}_1, \ldots, \mathbf{u}_M) \,|\, \Phi(\mathbf{u}_1, \ldots, \mathbf{u}_M; \mu) = \Phi^\infty\}$ denote the limit set. Using strict convexity of the objective, it can be shown that Algorithm 1 converges to a point $\mathbf{u}_1^\infty, \ldots, \mathbf{u}_M^\infty$ in S^∞. Because a coupled input constraint is present, the converged solution $\mathbf{u}_1^\infty, \ldots, \mathbf{u}_M^\infty$ may be different from the optimal centralized solution. Constraints and penalties on the rate of change of each subsystem's inputs can be included in the FC-MPC framework. These constraints and penalties result in additional local input constraints and extra terms in the controller objective function respectively. All established properties (feasibility, optimality and closed-loop stability) are binding.

In the second part of this series, we consider the output feedback case and establish feasibility, optimality and perturbed closed-loop stability properties for the cooperation-based distributed MPC formulation under intermediate termination. A framework for distributed state estimation, target calculation and disturbance modeling is described and the relevant properties are established.

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Appendices

A Preliminaries

Proof for Lemma 1. It follows from compactness of $\mathcal{X}$ and continuity of the linear mapping $\mathcal{A}(\cdot)$ that the set $\mathcal{B}$ is compact.

Let $\mathcal{A} = V\Sigma W'$ denote a singular value decomposition of $\mathcal{A}$. Also, let $r = \mathrm{rank}(\mathcal{A})$. Therefore,

$$\Sigma W' x = V' b$$

If $r < m$ then a necessary and sufficient condition for the system $\mathcal{A}x = b$ to be solvable is that the last $m - r$ left singular vectors are orthogonal to b. If $V = [v_1, v_2, \ldots, v_m]$, $W = [w_1, w_2, \ldots, w_n]$ and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots)$ then

$$\overline{x}(b) = \sum_{i=1}^{r} \frac{{v_i}' b}{\sigma_i} w_i$$

is a solution to the system $\mathcal{A}x = b$ with minimum l_2 norm [16, p. 429].

Since $0 \in \mathcal{X}$, there exists a $\varepsilon > 0$ such that $\overline{x}(b) \in \mathcal{X}$ for all $b \in B_\varepsilon(0)$

For $b \in B_\varepsilon(0)$, choose

$$K_1 = \sum_{i=1}^{r} \frac{\|v_i\| \|w_i\|}{\sigma_i}$$

This gives $\|\overline{x}(b)\| \le K_1 \|b\|, \mathcal{A}\overline{x}(b) = b, \overline{x}(b) \in \mathcal{X}$ with K_1 independent of the choice of $b \in B_\varepsilon(0)$.

Define

$$B \setminus B_\varepsilon(0) = \{b \mid b \in \mathcal{B}, \|b\| \ge \varepsilon\}$$

Compactness of $\mathcal{X}$ implies $\exists\, R > 0$ such that $\|x\| \le R,\ \forall\, x \in \mathcal{X}$. Therefore,

$$\|x\| \le \frac{R}{\varepsilon} \|b\|,\ \forall\, b \in B \setminus B_\varepsilon(0), x \in \mathcal{X}$$

The choice $K = \max(K_1, \frac{R}{\varepsilon})$ gives

$$\|\overline{x}(b)\| \le K \|b\|,\ \forall\, b \in \mathcal{B}$$

□

B Closed-loop properties of FC-MPC under state feedback

Proof for Lemma 6. The proof is by induction. At time $k = 0$, the FC-MPC algorithm is initialized with the input sequence $u_i(k+l|k) = 0,\ l \geq 0,\ \forall\ i \in \mathbb{I}_M$. Hence $J_N^0(\mu(0)) = \widetilde{J}_N(\mu(0))$. We know from Lemma 6 that $J_N^{p(0)}(\mu(0)) \leq J_N^0(\mu(0)) = \widetilde{J}_N(\mu(0))$. The relationship (Equation (13)), therefore, holds at $k = 0$. At time $k = 1$, we have

$$\begin{aligned} J_N^{p(1)}(\mu(1)) \leq J_N^0(\mu(1)) = J_N^{p(0)}(\mu(0)) - \sum_{i=1}^{M} w_i L_i(x_i(0), u_i^{p(0)}(0)) \\ \leq \widetilde{J}_N(\mu(0)) - \sum_{i=1}^{M} w_i L_i(x_i(0), 0) \\ \leq \widetilde{J}_N(\mu(0)) \end{aligned}$$

Hence the relationship in Equation (13) is true for $k = 1$. Assume now that the result is true for some time $k > 1$. At time $k + 1$,

$$\begin{aligned} J_N^{p(k+1)}(\mu(k+1) \leq J_N^0(\mu(k+1)) = J_N^{p(k)}(\mu(k)) - \sum_{i=1}^{M} w_i L_i \left(x_i(k), u_i^{p(k)}(k) \right) \\ \leq J_N^{p(k)}\left(\mu(k)\right) - \sum_{i=1}^{M} w_i L_i \left(x_i(k), 0\right) \\ \leq \widetilde{J}_N(\mu(0)) - \sum_{j=0}^{k} \sum_{i=1}^{M} w_i L_i(x_i(j), 0) \\ \leq \widetilde{J}_N(\mu(0)) \end{aligned}$$

The result is, therefore, true for all $k \geq 0$, as claimed. □

Proof for Theorem 1. To prove exponential stability, we use the value function $J_N^{p(k)}(\mu(k))$ as a candidate Lyapunov function. We need to show [39, p. 267] there exists constants $a, b, c > 0$ such that

$$a \sum_{i=1}^{M} \|x_i(k)\|^2 \leq J_N^p(\mu(k)) \leq b \sum_{i=1}^{M} \|x_i(k)\|^2 \tag{17a}$$

$$\Delta J_N^p(\mu(k)) \leq -c \sum_{i=1}^{M} \|x_i(k)\|^2 \tag{17b}$$

in which

$$\Delta J_N^{p(k)}(\mu(k)) = J_N^{p(k+1)}\left(\mu(k+1)\right) - J_N^{p(k)}\left(\mu(k)\right)$$

Let $\mathbf{x}_i^p = [x_i^p(\mu, 1)', x_i^p(\mu, 2)', \ldots]'$ denote the state trajectory for subsystem $i \in \mathbb{I}_M$ generated by the input trajectories $\mathbf{u}_1^p(\mu), \ldots, \mathbf{u}_M^p(\mu)$ obtained after $p \in \mathbb{I}_+$ Algorithm 1 iterates and initial state μ. Rewriting the cooperation-based cost function in terms of the calculated state and input trajectories, we have

$$J_N^{p(k)}(\mu(k)) = \sum_{i=1}^{M} w_i \left[\sum_{l=0}^{N-1} L_i(x_i^{p(k)}(\mu(k), l), u_i^{p(k)}(\mu(k), l)) + \frac{1}{2} \| x_i^{p(k)}(\mu(k), N) \|_{\overline{Q}_i}^2 \right]$$

in which Q_i, R_i and $\overline{Q}_i$ are all positive definite and $x_i^{p(k)}(\mu(k), 0) = x_i(k), i \in \mathbb{I}_M$.

Because $Q_i > 0$, there exists an $a > 0$ such that $a \sum_{i=1}^{M} \|x_i(k)\|^2 \leq J_N^{p(k)}(\mu(k))$. One possible choice is $a = \min_{i \in \mathbb{I}_M} \frac{1}{2} w_i \lambda_{\text{min}}(Q_i)$. From

$$\Delta J_N^{p(k)}(\mu(k)) \leq -\sum_{i=1}^{M} w_i L_i \left(x_i(k), u_i^{p(k)}(\mu(k), 0) \right) \leq -\sum_{i=1}^{M} w_i \frac{1}{2} x_i(k)' Q_i x_i(k),$$

there exists $c > 0$ such that $\Delta J_N^{p(k)}(\mu(k)) \leq -c \sum_{i=1}^{M} \|x_i(k)\|^2$. One possible choice for c is $c = \min_{i \in \mathbb{I}_M} \frac{1}{2} w_i \lambda_{\text{min}}(Q_i)$.

At time $k = 0$, each subsystem's FC-MPC optimization is initialized with the zero input trajectory. Using the definition of $\overline{Q}_i$ and Lemma 4, we have $J_N^{p(0)}(\mu(0)) \leq \sigma \sum_{i=1}^{M} \|x_i(0)\|^2$, in which $0 < \max_{i \in \mathbb{I}_M} w_i \lambda_{\max}(\overline{Q}_i) \leq \sigma$. Since $0 \in \text{int}\,(\Omega_1 \times \ldots \Omega_M)$ and the origin is Lyapunov stable and attractive with the cost relationship given in Lemma 6, there exists $\varepsilon > 0$ such that the input constraints remain inactive in each subsystem's FC-MPC optimization for any $\mu \in B_\varepsilon(0)$. From Remark 1, there exists $\rho > 0$ such that $\|\overline{\mathbf{u}}_i^p(\mu)\| \leq \sqrt{\rho} \|\mu\|, \forall\, \mu \in B_\varepsilon(0), 0 < p \leq p^*,\ i \in \mathbb{I}_M$. Using the definition of the norm operator on μ and squaring, we have $\|\overline{\mathbf{u}}_i^p(\mu)\|^2 \leq \rho \sum_{i=1}^{M} \|x_i\|^2$. Since $\Omega_i,\ \forall\, i \in \mathbb{I}_M$ is compact, a constant $\mathcal{Z} > 0$ exists satisfying $\|u_i\| \leq \sqrt{\mathcal{Z}},\ \forall\, u_i \in \Omega_i$ and all $i \in \mathbb{I}_M$. For $\|\mu\| > \varepsilon$, we have $\|u_i\| \leq \frac{\sqrt{\mathcal{Z}}}{\varepsilon} \|\mu\|$. Choose $K = \max(\rho, \frac{\mathcal{Z}}{\varepsilon^2}, \sigma) > 0$. The constant K is independent of x_i and $\|u_i^p(\mu, j)\|^2 \leq K \sum_{i=1}^{M} \|x_i\|^2,\ \forall\, i \in \mathbb{I}_M$, $j \geq 0$ and all $0 < p \leq p^*$.

Using stability of $A_i,\ i \in \mathbb{I}_M$ and [16, 5.6.13, p. 299], there exists $\overline{c} > 0$ and $\max_{i \in \mathbb{I}_M} \lambda_{\max}(A_i) \leq \lambda < 1$ such that $\|A_i^j\| \leq \overline{c} \lambda^j,\ \forall\, i \in \mathbb{I}_M, j \geq 0$. For any $i \in \mathbb{I}_M$ and $0 \leq l \leq N$, therefore,

$$\begin{aligned}
\|x_i^{p(k)}(\mu(k),l)\| &\leq \|A_i^l\|\|x_i(k)\| + \sum_{j=0}^{l-1} \|A_i^{l-1-j}\| \Big[\|B_i\|\|u_i^{p(k)}(\mu(k),j)\| \\
&\qquad + \sum_{s\neq i} \|W_{is}\|\|u_j^{p(k)}(\mu(k),j)\| \Big] \\
&\leq \overline{c}\lambda^l \|x_i(k)\| + \sum_{j=0}^{l-1} \overline{c}\lambda^{l-1-j}\gamma\sqrt{K} \left(\sum_{i=1}^{M} \|x_i(k)\|^2 \right)^{1/2} \\
&\leq \overline{c}\left(\lambda^l + \frac{\gamma\sqrt{K}}{1-\lambda}\right)\left(\sum_{i=1}^{M} \|x_i(k)\|^2 \right)^{1/2} \\
&\leq \sqrt{\Gamma}\left(\sum_{i=1}^{M} \|x_i(k)\|^2 \right)^{1/2}
\end{aligned}$$

in which $\gamma = \max_{i\in\mathbb{I}_M} \|B_i\| + \sum_{s\neq i}^{M} \|W_{is}\|$ and $\Gamma = \overline{c}^2 \left(1 + \frac{\gamma\sqrt{K}}{1-\lambda}\right)^2$. Hence,

$$\begin{aligned}
J_N^{p(k)}(\mu(k)) &\leq \sum_{i=1}^{M} w_i \Big[\frac{1}{2}\sum_{j=0}^{N-1} \lambda_{\max}(Q_i)\|x_i^{p(k)}(\mu(k),j)\|^2 + \lambda_{\max}(R_i)\|u_i^{p(k)}(\mu(k),j)\|^2 \\
&\qquad + \frac{1}{2}\lambda_{\max}(\overline{Q}_i)\|x_i^{p(k)}(\mu(k),N)\|^2 \Big] \\
&\leq \frac{1}{2}\sum_{i=1}^{M} w_i \left[\sum_{j=0}^{N-1} (\lambda_{\max}(Q_i)\Gamma + \lambda_{\max}(R_i)K) + \lambda_{\max}(\overline{Q}_i)\Gamma \right] \sum_{i=1}^{M} \|x_i(k)\|^2 \\
&= b\sum_{i=1}^{M} \|x_i(k)\|^2
\end{aligned}$$

in which the positive constant $b = \frac{1}{2}\sum_{i=1}^{M} w_i \left[N(\lambda_{\max}(Q_i)\Gamma + \lambda_{\max}(R_i)K) + \lambda_{\max}(\overline{Q}_i)\Gamma \right]$. □

Lemma 7. *Let $\Omega_i,\ i \in \mathbb{I}_M$ be specified in terms of a collection of linear inequalities. For each $i \in \mathbb{I}_M$, consider the FC-MPC optimization problem of Equation* (9) *with a terminal state constraint $U_{u_i}{}' \left(\mathcal{C}_N(A_i,B_i)\,\overline{\mathbf{u}}_i + A_i^N x_i(k)\right) = 0$. Let $B_\varepsilon(0),\ \varepsilon > 0$ be defined such that the input inequality constraints in each FC-MPC optimization problem and initialization QP (Equation* (14)*) remain inactive for $\mu \in B_\varepsilon(0)$. Let Assumptions 1, 4 and 7 hold. The input trajectory $\overline{\mathbf{u}}_i^p(\mu),\ i \in \mathbb{I}_M$ generated by Algorithm 1 is a Lipschitz continuous function of μ for all $p \in \mathbb{I}_+, p \leq p^*$.*

Proof. Since $0 \in \text{int}(\Omega_1 \times \cdots \times \Omega_M)$ and the distributed MPC control law is stable and attractive, an $\varepsilon > 0$ exists. From Assumptions 1 and 4, $(A_i, B_i),\ i \in \mathbb{I}_M$ is stabilizable. Because (A_i, B_i) is stabilizable and U_{u_i} is obtained from a Schur decomposition, the rows of $U_{u_i}{}'\mathcal{C}_N(A_i, B_i)$ are independent. Consider $\mu \in B_\varepsilon(0)$. Because all input inequality constraints are inactive and the rows of $U_{u_i}{}'\mathcal{C}_N(A_i, B_i)$ are independent, it follows that the active constraints are independent.

In the FC-MPC optimization problem of Equation (9) (with the terminal state constraint), $\mathcal{R}_i > 0$. The solution to the FC-MPC optimization problem is, therefore, unique. The parameters that vary in the data are μ and the input trajectories Δ_{-i}^{p-1}, in which $\Delta_{-i}^{p-1} = \overline{\mathbf{u}}_1^{p-1}, \ldots, \overline{\mathbf{u}}_{i-1}^{p-1}, \overline{\mathbf{u}}_{i+1}^{p-1}, \ldots, \overline{\mathbf{u}}_M^{p-1}$. Let $\overline{\mathbf{u}}_i^{*(p)}(\mu; \Delta_{-i}^{p-1}(\mu))$ represent the solution to the FC-MPC optimization problem at iterate p and system state μ. Also, let $\zeta = [z_1, z_2, \ldots, z_M]$. Since the set of active constraints is linearly independent, $\exists\ \rho < \infty$ such that [14, Theorem 3.1],

$$\|\overline{\mathbf{u}}_i^{*(p)}(\mu; \Delta_{-i}^{p-1}(\mu)) - \overline{\mathbf{u}}_i^{*(p)}(\zeta; \Delta_{-i}^{p-1}(\zeta)\| \leq \rho \left(\|\mu - \zeta\|^2 + \sum_{j \neq i}^{M} \|\overline{\mathbf{u}}_j^{p-1}(\mu) - \overline{\mathbf{u}}_j^{p-1}(\zeta)\|^2 \right)^{1/2}$$

From Algorithm 1, we have

$$\begin{aligned} \|\overline{\mathbf{u}}_i^p(\mu) - \overline{\mathbf{u}}_i^p(\zeta)\| &\leq w_i \|\overline{\mathbf{u}}_i^{*(p)}(\mu;\ \cdot) - \overline{\mathbf{u}}_i^{*(p)}(\zeta;\ \cdot)\| \\ &\quad + (1 - w_i) \|\overline{\mathbf{u}}_i^{p-1}(\mu) - \overline{\mathbf{u}}_i^{p-1}(\zeta)\| \\ &\leq \rho w_i \left(\|\mu - \zeta\|^2 + \sum_{j \neq i}^{M} \|\overline{\mathbf{u}}_j^{p-1}(\mu) - \overline{\mathbf{u}}_j^{p-1}(\zeta)\|^2 \right)^{1/2} \\ &\quad + (1 - w_i) \|\overline{\mathbf{u}}_i^{p-1}(\mu) - \overline{\mathbf{u}}_i^{p-1}(\zeta)\|,\ p \in \mathbb{I}_+ \end{aligned} \tag{18}$$

It follows from Equation (18) that if $\overline{\mathbf{u}}_i^{p-1}(\mu)$ is Lipschitz continuous w.r.t μ for all $i \in \mathbb{I}_M$ then $\overline{\mathbf{u}}_i^p(\mu)$ is Lipschitz continuous w.r.t μ.

For $k > 0$, we have (Equation (12))

$$\overline{\mathbf{u}}_i^0(k) = \left[u_i^{p(k-1)}(\mu(k-1), 1)', \ldots, u_i^{p(k-1)}(\mu(k-1), N-1)', 0 \right],$$

independent of the current system state μ (hence Lipschitz in μ by causality of the models). To complete the proof, we need to show that the initialization strategy (the solution to the QP, Equation (14)) is Lipschitz continuous in the state.

Consider two sets of initial subsystem states $\mu(0), \zeta(0) \in \mathbb{D}_C$ containing the initial subsystem states $x_i(0)$ and $z_i(0)$, $i \in \mathbb{I}_M$, respectively. Let the solution to the initialization QP (Equation (14)) for the two initial states be $\overline{\mathbf{v}}_i^*(x_i(0))$ and $\overline{\mathbf{v}}_i^*(z_i(0))$, respectively. Because the input constraints are inactive and the rows of $U_{u_i}{}'\mathcal{C}_N(A_i, B_i)$ are independent, the active constraints for the initialization QP are independent for $\mu \in B_\varepsilon(0)$. From [14, Theorem 3.1], $\exists\ \rho < \infty$ satisfying $\|\overline{\mathbf{v}}_i^*(x_i(0)) - \overline{\mathbf{v}}_i^*(z_i(0))\| \leq \rho \|x_i(0) - z_i(0)\|$. We have $\overline{\mathbf{u}}_i^0(\cdot) = \overline{\mathbf{v}}_i^*(x_i(0)),\ i \in \mathbb{I}_M$. This gives $\|\overline{\mathbf{u}}_i^0(\mu(0)) - \overline{\mathbf{u}}_i^0(\zeta(0))\| = \|\overline{\mathbf{v}}_i^*(x_i(0)) - \overline{\mathbf{v}}_i^*(z_i(0))\| \leq \rho \|x_i(0) - z_i(0)\| \leq \rho \|\mu(0) - \zeta(0)\|, \forall\ i \in \mathbb{I}_M$.

Thus, either initialization is Lipschitz in the current system state μ. Hence, $\overline{\mathbf{u}}_i^1(\mu)$ is Lipschitz continuous in μ. Subsequently, by induction, $\overline{\mathbf{u}}_i^p(\mu)$ is Lipschitz continuous in

μ for all $p \in \mathbb{I}_+$. For $p_{\max}(k) \leq p^* < \infty$ for all $k \geq 0$, a global Lipschitz constant can be estimated. By definition, $\mathbf{u}_i^p(\mu) = [\overline{\mathbf{u}}_i^p(\mu)', 0, 0, \ldots]'$, $i \in \mathbb{I}_M$. Hence, $\mathbf{u}_i^p(\mu)$ is a Lipschitz continuous function of μ for all $p \in \mathbb{I}_+$, $p \leq p_{\max}$. □

Proof for Theorem 2. To show exponential stability for the set of initial subsystem states $\mu(k) \in \mathbb{D}_C$, we need to show that there exists positive constants a, b, c satisfying Equations (17a) and (17b). Determination of constants a and c closely follows the argument presented in the proof for Theorem 1. To complete the proof, we need to show that a constant $b > 0$ exists such that

$$J_N^{p(k)}(\mu(k)) \leq b \sum_{i=1}^{M} \|x_i(k)\|^2$$

Let $\mathbf{x}_i^p = [x_i^p(\mu, 1)', x_i^p(\mu, 2)', \ldots]'$ denote the state trajectory for subsystem $i \in \mathbb{I}_M$ generated by the input trajectories $\mathbf{u}_1^p(\mu), \ldots, \mathbf{u}_M^p(\mu)$ obtained after $p \in \mathbb{I}_+$ (Algorithm 1) iterates and initial state μ. Let $x_i^{p(k)}(\mu(k), 0) = x_i(k)$, $i \in \mathbb{I}_M$. At time $k = 0$, we have $\mu(0) \in \mathbb{D}_C$. From the definition of the initialization QP (Equation (14)) and Lemma 1, there exists constant $K_{u_i} > 0$ independent of x_i such that the N input sequence $\widetilde{\mathbf{u}}_i(0)' = [\widetilde{u}_i(0|0)', \widetilde{u}_i(1|0)', \ldots, \widetilde{u}_i(N-1|0)']$ obtained as the solution to Equation (14) satisfies

$$\|\widetilde{u}_i(l|0)\| \leq \sqrt{K_{u_i}} \|x_i(0)\| \leq \sqrt{K_{u_i}} \|\mu(0)\|, 0 \leq l \leq N-1, \ i \in \mathbb{I}_M.$$

Let $K_u = \max_{i \in \mathbb{I}_M} K_{u_i}$. Since $0 \in \text{int}(\Omega_1 \times \ldots \times \Omega_M)$ and the origin is Lyapunov stable and attractive with the cost relationship given in Section 7.2, there exists an $\varepsilon_1 > 0$ such that all the input inequality constraints in the FC-MPC optimization for each subsystem remain inactive for any $\mu \in B_{\varepsilon_1}(0)$. Similarly, choose $\varepsilon_2 > 0$ such that the minimum l_2 norm solution is feasible (and hence optimal) for the initialization QP (Equation (14)) for all $i \in \mathbb{I}_M$ and any $\mu \in B_{\varepsilon_2}(0)$. Feasibility of the minimum l_2 norm solution implies none of the input inequality constraints are active. Pick $\varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0$. For $\mu \in B_\varepsilon(0)$, the only active constraint, for each subsystem $i \in \mathbb{I}_M$, is the terminal equality constraint ${U_{u_i}}' \left(\mathcal{C}_N(A_i, B_i)\, \overline{\mathbf{u}}_i + A_i^N x_i(k)\right) = 0$. In the above constraint, U_{u_i} is obtained from a Schur decomposition of A_i and (A_i, B_i) is stabilizable. The rows of ${U_{u_i}}'\mathcal{C}_N(A_i, B_i)$ are, therefore, linearly independent. From Lemma 7, $\mathbf{u}_i(\cdot)$ is Lipschitz continuous in μ for $\mu \in B_\varepsilon(0)$. Hence, there exists $\rho > 0$ such that $\|\mathbf{u}_i^p(\mu)\|^2 \leq \rho \sum_{i=1}^M \|x_i\|^2$, $\forall\, 0 < p \leq p^*$, $i \in \mathbb{I}_M$ [‖]. Using arguments identical to those described in the proof for Theorem 1, we have $K = \max(\rho, \frac{\mathcal{Z}}{\varepsilon}^2, K_u)$ where $K > 0$ and independent of $x_i, i \in \mathbb{I}_M$ such that $\|u_i^p(\mu, j)\|^2 \leq K \sum_{i=1}^M \|x_i\|^2$, $\forall\, i \in \mathbb{I}_M$, $j \geq 0$, and all $0 < p \leq p^*$.

[‖] The details are available in the proof for Theorem 1 and are, therefore, omitted.

Define $\mathcal{A}_i = \max_{0 \leq j \leq N} \|A_i^j\|$. For any $i \in \mathbb{I}_M$ and $0 \leq l \leq N$

$$\begin{aligned}
\|x_i^{p(k)}(\mu(k), l)\| &= \|A_i^l x_i^{p(k)}(\mu(k), 0) + \sum_{j=0}^{l-1} A_i^{l-1-j} B_i u_i^{p(k)}(\mu(k), j) \\
&\quad + \sum_{s \neq i} \sum_{j=0}^{l-1} A_i^{l-1-j} W_{is} u_s^{p(k)}(\mu(k), j)\| \\
&\leq \|A_i^l\| \|x_i(k)\| + \sum_{j=0}^{l-1} \|A_i^{l-1-j}\| \left[\|B_i\| + \sum_{s \neq i} \|W_{is}\| \right] \sqrt{K} \left(\sum_{i=1}^{M} \|x_i(k)\|^2 \right)^{1/2} \\
&\leq \mathcal{A}_i \|x_i(k)\| + \mathcal{A}_i \sum_{j=0}^{l-1} \gamma \sqrt{K} \left(\sum_{i=1}^{M} \|x_i(k)\|^2 \right)^{1/2} \\
&\leq \mathcal{A}_i \left(1 + \gamma N \sqrt{K} \right) \left(\sum_{i=1}^{M} \|x_i(k)\|^2 \right)^{1/2} \\
&= \sqrt{\Gamma_i} \left(\sum_{i=1}^{M} \|x_i(k)\|^2 \right)^{1/2}
\end{aligned}$$

in which $\gamma = \max_{i \in \mathbb{I}_M} \|B_i\| + \sum_{s \neq i}^{M} \|W_{is}\|$ and $\Gamma_i = \mathcal{A}_i \left(1 + \gamma N \sqrt{K} \right)^2$.

Define Σ_i to be the solution to the Lyapunov equation $A_{s_i}{}' \Sigma_i A_{s_i} - \Sigma_i = -U_{s_i}{}' Q_i U_{s_i}$ and let $\overline{Q}_i = U_{s_i} \Sigma_i U_{s_i}{}'$. The infinite sum $\sum_{i=1}^{M} w_i \sum_{j=N}^{\infty} L_i(A_i^l x_i^{p(k)}(\mu(k), j), 0)$ subject to the terminal state constraint $U_{u_i}{}' x_i^{p(k)}(\mu(k), N) = 0$ is equal to

$$\sum_{i=1}^{M} w_i \frac{1}{2} x_i^{p(k)}(\mu(k), N)' \overline{Q}_i x_i^{p(k)}(\mu(k), N).$$

Therefore,

$$
\begin{aligned}
J_N^{p(k)}(\mu(k)) &= \sum_{i=1}^{M} w_i \Big[\sum_{j=0}^{N-1} L_i \left(x_i^{p(k)}(\mu(k), j), u_i^{p(k)}(\mu(k), j) \right) \\
&\qquad + \frac{1}{2} x_i^{p(k)}(\mu(k), N)' \overline{Q}_i x_i^{p(k)}(\mu(k), N) \Big] \\
&\leq \sum_{i=1}^{M} w_i \Big[\frac{1}{2} \sum_{j=0}^{N-1} \lambda_{\max}(Q_i) \| x_i^{p(k)}(\mu(k), j) \|^2 + \lambda_{\max}(R_i) \| u_i^{p(k)}(\mu(k), j) \|^2 \\
&\qquad + \frac{1}{2} \lambda_{\max}(\overline{Q}_i) \| x_i^{p(k)}(\mu(k), N) \|^2 \Big] \\
&\leq \frac{1}{2} \sum_{i=1}^{M} w_i \left[\sum_{j=0}^{N-1} (\lambda_{\max}(Q_i)\Gamma_i + \lambda_{\max}(R_i) K) + \lambda_{\max}(\overline{Q}_i)\Gamma_i \right] \sum_{i=1}^{M} \| x_i(k) \|^2 \\
&\leq \frac{1}{2} \sum_{i=1}^{M} w_i \left[N\lambda_{\max}(Q_i)\Gamma_i + N\lambda_{\max}(R_i) K + \lambda_{\max}(\overline{Q}_i)\Gamma_i \right] \sum_{i=1}^{M} \| x_i(k) \|^2 \\
&\leq b \sum_{i=1}^{M} \| x_i(k) \|^2
\end{aligned}
$$

in which the constant b is independent of the initialization strategy used for the subsystem input trajectories and selected such that

$$
0 \leq \frac{1}{2} \sum_{i=1}^{M} w_i \left[N\lambda_{\max}(Q_i)\Gamma_i + N\lambda_{\max}(R_i) K + \lambda_{\max}(\overline{Q}_i)\Gamma_i \right] \leq b
$$

Hence, the closed-loop system is exponentially stable, as claimed. □