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A convergence result for nonmonotonic cost sequences

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Consider the infinite sequence $\{\Phi_k\}$ that is assumed to satisfy

$$\Phi_{k+j} \leq \Phi_k + S_{k,j}, \qquad \Phi_k \geq 0 \tag{1}$$

for all $k, j \in \mathbb{I}_+ := \{0, 1, 2, \ldots\}$. We assume:

A1: For all $k \in \mathbb{I}_+$, for all $\varepsilon > 0$, there exists an integer $I(k, \varepsilon)$ such that $S_{k,j} \leq \varepsilon$ for all $j \geq I(k, \varepsilon)$.

Proposition 1 *Suppose **A1** is satisfied. Then, for all $\Phi_0 \geq 0$, the infinite sequence $\{\Phi_k \mid k \in \mathbb{I}_+\}$ is bounded.*

Proof: Choose any $\varepsilon > 0$. By **A1**, there exists a positive integer $I(0, \varepsilon)$ such that $S_{0,j} \leq \varepsilon$ for all $j \geq I(0, \varepsilon)$. Hence

$$\Phi_k \leq \Phi_0 + \varepsilon \qquad \text{for all } k \geq I(0, \varepsilon).$$

Also, there clearly exists a $\widehat{\Phi}$ such that

$$\Phi_k \leq \widehat{\Phi} \qquad \text{for all } k \in \{0, 1, \ldots, I(0, \varepsilon)\}.$$

Hence

$$\Phi_k \leq \widetilde{\Phi} := \max\{\widehat{\Phi}, \Phi_0 + \varepsilon\} \quad \text{for all } k \in \mathbb{I}_+.$$

so that $\Phi_k \in [0, \widetilde{\Phi}]$ for all $k \in \mathbb{I}_+$. ∎

Proposition 2 *Suppose **A1** is satisfied. Then, for all $\Phi_0 \geq 0$, the infinite sequence $\{\Phi_k \mid k \in \mathbb{I}_+\}$ converges to a $\Phi^* \in [0, \widetilde{\Phi}]$.*

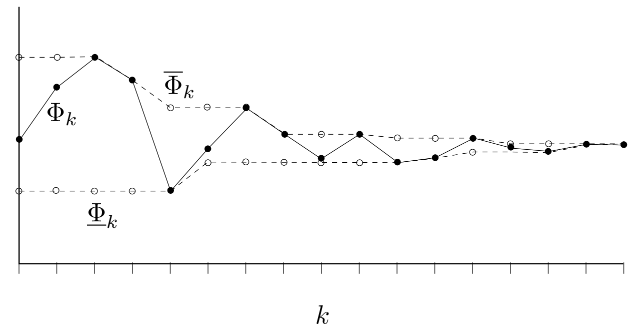

Figure 1: Nonmonotonic cost sequence with its upper and lower bounding sequences.

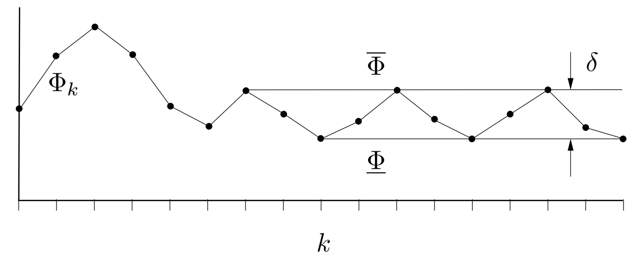

Figure 2: Nonconvergent cost sequence with $\overline{\Phi} - \underline{\Phi} = \delta > 0$.

Proof: Consider the two bounding sequences $\{\overline{\Phi}_k\}$ and $\{\underline{\Phi}_k\}$ defined by

$$\overline{\Phi}_k := \sup_{j \geq k} \Phi_j \qquad \underline{\Phi}_k := \inf_{j \geq k} \Phi_j$$

These sequences are illustrated in Figure 1. By Proposition 1, the sequence $\{\Phi_k\}$ is bounded, so that both $\overline{\Phi}_k$ and $\underline{\Phi}_k$ are finite for all $k \geq 0$. In fact, both lie in the interval $[0, \widetilde{\Phi}]$. The sequence $\{\overline{\Phi}_k\}$ is nonincreasing and bounded from below (by 0) so that it converges to $\overline{\Phi} = \limsup\{\Phi_k\} \in [0, \widetilde{\Phi}]$. Similarly the sequence $\{\underline{\Phi}_k\}$ is nondecreasing and bounded from above (by $\widetilde{\Phi}$) so that it converges to $\underline{\Phi} = \liminf\{\Phi_k\} \in [0, \widetilde{\Phi}]$ with $\underline{\Phi} \leq \overline{\Phi}$. Both $\overline{\Phi}$ and $\underline{\Phi}$ are accumulation points of the sequence $\{\Phi_k\}$. In fact, all the accumulation points of $\{\Phi_k\}$ lie in the interval $[\underline{\Phi}, \overline{\Phi}] \subset [0, \widetilde{\Phi}]$.

Suppose, contrary to what is to be proven, that $\underline{\Phi} \neq \overline{\Phi}$, and let $\delta := \overline{\Phi} - \underline{\Phi} > 0$ as depicted in Figure 2. Because $\underline{\Phi}$ is an accumulation point of $\{\Phi_k\}$, there exists a positive integer I_1 such that

$$\Phi_{I_1} \leq \underline{\Phi} + \delta/4$$

By assumption **A1**, there exists an integer I_2 such that $S_{I_1,j} \leq \delta/4$ for all $j \geq I_2$. Therefore, from Equation 1

$$\Phi_k \leq \Phi_{I_1} + \delta/4 \qquad \text{for all } k \geq I_1 + I_2$$

Combining the two preceding inequalities gives

$$\Phi_k \leq \underline{\Phi} + \delta/2 \qquad \text{for all } k \geq I_1 + I_2$$

Substituting the definition of δ in the last inequality gives

$$\Phi_k \leq \overline{\Phi} - \delta/2 \qquad \text{for all } k \geq I_1 + I_2$$

But this inequality contradicts the fact that $\overline{\Phi}$ is an accumulation point of $\{\Phi_k\}$. Hence the sequence $\{\Phi_k\}$ converges to a $\Phi^* = \underline{\Phi} = \overline{\Phi} \in [0, \widetilde{\Phi}]$. ∎