Consider the infinite sequence \( \{\Phi_k\} \) that is assumed to satisfy

\[
\Phi_{k+j} \leq \Phi_k + S_{k,j}, \quad \Phi_k \geq 0
\]

for all \( k, j \in \mathbb{I}_+ := \{0, 1, 2, \ldots\} \). We assume:

**A1**: For all \( k \in \mathbb{I}_+ \), for all \( \varepsilon > 0 \), there exists an integer \( I(k, \varepsilon) \) such that \( S_{k,j} \leq \varepsilon \) for all \( j \geq I(k, \varepsilon) \).

**Proposition 1** Suppose **A1** is satisfied. Then, for all \( \Phi_0 \geq 0 \), the infinite sequence \( \{\Phi_k | k \in \mathbb{I}_+\} \) is bounded.

**Proof**: Choose any \( \varepsilon > 0 \). By **A1**, there exists a positive integer \( I(0, \varepsilon) \) such that \( S_{0,j} \leq \varepsilon \) for all \( j \geq I(0, \varepsilon) \). Hence

\[
\Phi_k \leq \Phi_0 + \varepsilon \quad \text{for all } k \geq I(0, \varepsilon).
\]

Also, there clearly exists a \( \tilde{\Phi} \) such that

\[
\Phi_k \leq \tilde{\Phi} \quad \text{for all } k \in \{0, 1, \ldots, I(0, \varepsilon)\}.
\]

Hence

\[
\Phi_k \leq \tilde{\Phi} := \max\{\tilde{\Phi}, \Phi_0 + \varepsilon\} \quad \text{for all } k \in \mathbb{I}_+.
\]

so that \( \Phi_k \in [0, \tilde{\Phi}] \) for all \( k \in \mathbb{I}_+ \).

**Proposition 2** Suppose **A1** is satisfied. Then, for all \( \Phi_0 \geq 0 \), the infinite sequence \( \{\Phi_k | k \in \mathbb{I}_+\} \) converges to a \( \Phi^* \in [0, \tilde{\Phi}] \).
Figure 1: Nonmonotonic cost sequence with its upper and lower bounding sequences.

Figure 2: Nonconvergent cost sequence with $\Phi - \Phi = \delta > 0$.

**Proof:** Consider the two bounding sequences $\{\Phi_k\}$ and $\{\Phi_k\}$ defined by

$$
\Phi_k := \sup_{j \geq k} \Phi_j \quad \Phi_k := \inf_{j \geq k} \Phi_j
$$

These sequences are illustrated in Figure 1. By Proposition 1, the sequence $\{\Phi_k\}$ is bounded, so that both $\Phi_k$ and $\Phi_k$ are finite for all $k \geq 0$. In fact, both lie in the interval $[0, \Phi]$. The sequence $\{\Phi_k\}$ is nonincreasing and bounded from below (by 0) so that it converges to $\Phi = \limsup \{\Phi_k\} \in [0, \Phi]$. Similarly the sequence $\{\Phi_k\}$ is nondecreasing and bounded from above (by $\Phi$) so that it converges to $\Phi = \liminf \{\Phi_k\} \in [0, \Phi]$ with $\Phi \leq \Phi$. Both $\Phi$ and $\Phi$ are accumulation points of the sequence $\{\Phi_k\}$. In fact, all the accumulation points of $\{\Phi_k\}$ lie in the interval $[\Phi, \Phi] \subset [0, \Phi]$.

Suppose, contrary to what is to be proven, that $\Phi \neq \Phi$, and let $\delta := \Phi - \Phi > 0$ as depicted in Figure 2. Because $\Phi$ is an accumulation point of $\{\Phi_k\}$, there exists a positive integer $I_1$ such that

$$
\Phi_{I_1} \leq \Phi + \delta/4
$$

By assumption A1, there exists an integer $I_2$ such that $S_{I_1, j} \leq \delta/4$ for all $j \geq I_2$. Therefore, from Equation 1

$$
\Phi_k \leq \Phi_{I_1} + \delta/4 \quad \text{for all } k \geq I_1 + I_2
$$

Combining the two preceding inequalities gives

$$
\Phi_k \leq \Phi + \delta/2 \quad \text{for all } k \geq I_1 + I_2
$$
Substituting the definition of $\delta$ in the last inequality gives

$$\Phi_k \leq \overline{\Phi} - \delta/2 \quad \text{for all } k \geq I_1 + I_2$$

But this inequality contradicts the fact that $\overline{\Phi}$ is an accumulation point of $\{\Phi_k\}$. Hence the sequence $\{\Phi_k\}$ converges to a $\Phi^* = \Phi = \overline{\Phi} \in [0, \overline{\Phi}]$. ■