TWCCC ⋆ Texas – Wisconsin – California Control Consortium

Technical report number 2017–01

On the Stabilization of Neutrally Stable Linear Discrete Time Systems

Travis J. Arnold* and James B. Rawlings†
Department of Chemical and Biological Engineering
University of Wisconsin-Madison

September 9, 2017

1 Introduction

In this report we construct a stabilizing control law for neutrally stable linear discrete time systems. This is a well studied problem in the literature, particularly in the context where the system is subject to input saturation (Choi, 1999; Bao et al., 2000; Shi et al., 2003; Kim et al., 2004). In particular, Bao et al. (2000) and Shi et al. (2003) construct a linear control law $u = Kx$ such that the closed-loop transition matrix for the unconstrained system, $A + BK$, is stable. This is the result we present in this report.

The particular control law discussed is not of much practical interest—we acknowledge that in practice, using the linear quadratic regulator is generally the most practical control strategy. The purpose of the report is to provide more exposition in proving the result than is given in any of the aforementioned references. The result is also insightful because it draws on LaSalle's invariance principle, which is useful in many other contexts as well.

2 Preliminaries

We provide the following material such that this note may be reasonably self contained. All notation and definitions are consistent with those used in Rawlings and Mayne (2009, Appendix B) except where noted.

* `tjarnold@wisc.edu`
† `james.rawlings@wisc.edu`

2.1 System Notation

We consider systems that evolve in discrete time as

$$x^+ = f(x, u)$$

where the state is $x \in \mathbb{R}^n$ and the input is $u \in \mathbb{R}^m$. Let $\phi(k; x, \boldsymbol{u})$ denote the solution to the system at time k in response to the input sequence $\boldsymbol{u} = (u(0), u(1), \cdots)$ where the initial state is $x(0) = x$. This note is concerned with linear systems, which have the form

$$x^+ = Ax + Bu$$

where $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$.

2.2 Comparison Functions and Stability

Definition. A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if it is continuous, $\beta(\cdot, t)$ is a $\mathcal{K}$ function for each $t \geq 0$, and $\beta(s, \cdot)$ is nonincreasing with $\lim_{t\to\infty} \beta(s, t) = 0$ for each $s \geq 0$.

Definition. The origin is said to be *globally asymptotically stable* for $x^+ = f(x)$ if there exists a $\mathcal{KL}$ function β such that for each $x \in \mathbb{R}^n$ and $i \in \mathbb{I}_{\geq 0}$

$$|\phi(i; x)| \leq \beta(|x|, i)$$

Remark. For a linear system $x^+ = Ax$, the origin is globally asymptotically stable if and only if all of the eigenvalues of A are inside the unit circle.

2.3 LaSalle Invariance

Definition. The *distance* from a point $z \in \mathbb{R}^p$ to a set $C \subseteq \mathbb{R}^p$ is defined as

$$|z|_C = \inf_{x\in C} \|z - x\|_2$$

Definition. A sequence $z_k \in \mathbb{R}^p$ is said to *converge* to the set C if $|z_k|_C \to 0$.

Definition. Let $G \subseteq \mathbb{R}^n$. $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a *semi-Lyapunov function* of $x^+ = f(x)$ on G if V is continuous and $V(f(x)) - V(x) \leq 0$ for all $x \in G$.

Remark. The name semi-Lyapunov function is not standard. We employ it here to make explicit that a semi-Lyapunov function is different than a Lyapunov function as defined by Rawlings and Mayne (2009), who require $V(f(x)) - V(x) < 0$ for all $x \neq 0$ and do not require a Lyapunov function to be continuous.

Theorem 1 (LaSalle's invariance principle)**.** *Assume f is continuous and let V be a semi-Lyapunov function of $x^+ = f(x)$ on G. Define*

$$E = \{x : V(f(x)) - V(x) = 0, x \in \bar{G}\}$$

If (x_k) is a solution to $x^+ = f(x)$ that is bounded and in G for all $k \geq 0$, then $x_k \to E$.

This is Theorem 6.3 from LaSalle (1976), where a proof may be found. Note that the version of the theorem as given by LaSalle is slightly stronger than what is presented here; LaSalle says that (x_k) converges to a particular subset of E. However, for the example in this note convergence to E is what is important, so the version given here is sufficient.

3 Main Results

Theorem 2. *Suppose A is unitary, (A, B) is controllable, and $\kappa > 0$, $\kappa B^* B < 2I_m$. Then the origin is globally asymptotically stable for the linear control law $u(x) = Kx$ where $K = -\kappa B^* A$.*

Proof. The closed-loop system evolves as

$$x^+ = (A + BK)x$$

which has solution

$$\begin{aligned} x_k &= (A + BK)^k x_0 \\ &= (A - \kappa BB^* A)^k x_0 \end{aligned}$$

We want to prove that for arbitrary x_0, $x_k \to 0$. To this end, fix $x_0 \neq 0$ (if $x_0 = 0$ it follows immediately that $x_k \to 0$) and consider the function

$$V(x) = x^* x$$

and observe

$$\begin{aligned} V(x^+) - V(x) &= x^*(A + BK)^*(A + BK)x - x^* x \\ &= x^*(A^* A + A^* BK + K^* B^* A + K^* B^* BK - I_m)x \\ &= x^*(-2\kappa A^* BB^* A + \kappa^2 A^* BB^* BB^* A)x \\ &= x^*(\kappa A^* B(\kappa B^* B - 2I_m)B^* A)x \\ &= x^*(\kappa A^* BCB^* A)x \\ &= x^* Mx \end{aligned}$$

where $C = \kappa B^* B - 2I_m < 0$ and $M = \kappa A^* BCB^* A \leq 0$. Hence V is a semi-Lyapunov function for the closed-loop system on $\mathbb{R}^n$. The next step is to invoke LaSalle's invariance principle (LaSalle, 1976, Theorem 6.3), but to do so we need to prove that the closed-loop trajectory of the system is bounded. To this end, observe

$$\begin{aligned} V(x^+) &\leq V(x) \\ (x^+)^* x^+ &\leq x^* x \end{aligned}$$

which (by induction) implies $x_k^* x_k \leq x_0^* x_0$ for all k. In other words, for all k, $x_k \in X$ where

$$X = \{x : \|x\|_2 \leq \|x_0\|_2\}$$

So the closed-loop trajectory is bounded. Now, LaSalle's invariance principle implies that (x_k) must converge to the null space of M. In other words, $x_k \to Y$ where $Y = (\operatorname{null} M) \cap X$. Y is compact because $\operatorname{null} M$ is closed and X is compact.

Claim. If $x_0 \in \operatorname{null} M$ and $x_0 \neq 0$ then $x_\ell \notin \operatorname{null} M$ for some $\ell > 0$.

Assume the claim, and then assume towards a contradiction that $x_k \not\to 0$. Because X is compact, (x_k) has a subsequential limit, $\bar{x} \in \operatorname{null} M$. We may assume that $\bar{x} \neq 0$, because if 0 were the only subsequential limit of (x_k) then we would have $x_k \to 0$ (see Lemma 2 in the Appendix). By the claim, there exists ℓ such that $x^\star = (A + BK)^\ell \bar{x} \notin \operatorname{null} M$. Then

$$d = |x^\star|_{\operatorname{null} M} > 0$$

By continuity, there exists $\varepsilon > 0$ such that

$$\begin{aligned} \|x - \bar{x}\|_2 < \varepsilon &\implies \|(A + BK)^\ell x - x^\star\|_2 < d/2 \\ &\implies |(A + BK)^\ell x|_{\operatorname{null} M} > d/2 \end{aligned}$$

where the second implication follows from use of the triangle inequality. We use this fact as follows: because $\bar{x}$ is a subsequential limit of (x_k), there exists arbitrarily large N such that

$$\|x_N - \bar{x}\|_2 < \varepsilon \implies |x_{N+\ell}|_{\operatorname{null} M} > d/2$$

This implies $x_k \not\to M$ which is a contradiction, so we conclude $x_k \to 0$.

All that remains is to prove the claim. Towards a contradiction, assume that

$$x_0 \in \operatorname{null} M,\ x_0 \neq 0 \implies x_k \in \operatorname{null} M \text{ for all } k \geq 0$$

Pick arbitrary $x_0 \in \operatorname{null} M$, $x_0 \neq 0$. We will show by induction that for all $k \geq 1$, $B^* A^k x_0 = 0$ and $(A - \kappa BB^* A)^k x_0 = A^k x_0$. We start with $k = 1$:

$$\begin{aligned} Mx_0 &= 0 \\ \kappa A^* BCB^* Ax_0 &= 0 \\ x_0^* A^* BCB^* Ax_0 &= 0 \\ B^* Ax_0 &= 0 \end{aligned}$$

where the last equation follows because $C < 0$. We use this to obtain

$$\begin{aligned} (A - \kappa BB^* A)x_0 &= Ax_0 - \kappa BB^* Ax_0 \\ &= Ax_0 \end{aligned}$$

Now assume that the induction hypothesis holds for arbitrary k.

$$\begin{aligned} Mx_k &= 0 \\ M(A - \kappa BB^* A)^k x_0 &= 0 \\ MA^k x_0 &= 0 \\ \kappa A^* BCB^* A^{k+1} x_0 &= 0 \\ x_0^* (A^*)^{k+1} BCB^* A^{k+1} x_0 &= 0 \\ B^* A^{k+1} x_0 &= 0 \end{aligned}$$

where the last equation follows because $C < 0$. We use this to obtain

$$\begin{aligned}(A - \kappa BB^* A)^{k+1} x_0 &= (A - \kappa BB^* A)(A - \kappa BB^* A)^k x_0 \\ &= (A - \kappa BB^* A) A^k x_0 \\ &= A^{k+1} x_0 - \kappa BB^* A^{k+1} x_0 \\ &= A^{k+1} x_0\end{aligned}$$

This completes the induction. We now have the following equation:

$$\begin{bmatrix} B^* \\ B^* A \\ \vdots \\ B^* A^{n-1} \end{bmatrix} A x_0 = 0$$

From Lemma 1, (A^*, B) is controllable, so this implies that $Ax_0 = 0$. But $Ax_0 \neq 0$ because $x_0 \neq 0$ and A is invertible. So we have a contradiction. □

Theorem 3. *Suppose A is diagonalizable with all its eigenvalues on the unit circle. That is, A has eigenvalue decomposition $A = P^{-1} \Lambda P$ where Λ is diagonal and unitary. Suppose also that (A, B) is controllable and $\kappa > 0$, $\kappa B^* P^* P B < 2 I_m$. Then the origin is globally asymptotically stable for the linear control law $u(x) = Kx$ where $K = -\kappa B^* P^* \Lambda P$.*

Proof. First perform a change of coordinates using P:

$$\begin{aligned} x^+ &= Ax + Bu \\ x^+ &= P^{-1} \Lambda P x + Bu \\ P x^+ &= \Lambda P x + PBu \\ \tilde{x}^+ &= \Lambda \tilde{x} + PBu \end{aligned}$$

where $\tilde{x} = Px$. Next, observe

$$\begin{aligned} (A, B) \text{ controllable} &\iff \operatorname{rank}\big(\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}\big) = n \\ &\iff \operatorname{rank}\big(P \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}\big) = n \\ &\iff \operatorname{rank}\big(\begin{bmatrix} PB & PAB & \cdots & PA^{n-1}B \end{bmatrix}\big) = n \\ &\iff \operatorname{rank}\big(\begin{bmatrix} PB & \Lambda PB & \cdots & \Lambda^{n-1}PB \end{bmatrix}\big) = n \\ &\iff (\Lambda, PB) \text{ controllable} \end{aligned}$$

where the second equivalence follows because P is invertible. By Theorem 2, $u(\tilde{x}) = \tilde{K}\tilde{x}$ where $\tilde{K} = -\kappa B^* P^* \Lambda$ is a stabilizing linear control law for the transformed coordinates. The two sets of coordinates are related by an invertible matrix, so this control law must also be stabilizing for the original coordinates. In terms of the original coordinates, this control law is $u(x) = Kx$ where $K = -\kappa B^* P^* \Lambda P$. □

Definition. An eigenvalue of a square matrix A is said to be *semisimple* if its algebraic multiplicity is equal to its geometric multiplicity.

Remark. A square matrix is diagonalizable if and only if all of its eigenvalues are semisimple.

Definition. A matrix A is said to be *neutrally stable* if all of its eigenvalues lie on or inside the unit circle, and those on the unit circle are semisimple.

Theorem 4. *Suppose A is neutrally stable, with n_1 eigenvalues on the unit circle and n_2 eigenvalues inside the unit circle. Then there exists a similarity transformation of A*

$$A = M^{-1}\tilde{A}M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

where $\tilde{A}_{11} \in \mathbb{C}^{n_1 \times n_1}$ is diagonalizable ($\tilde{A}_{11} = P^{-1}\Lambda P$) as has all its eigenvalues on the unit circle and $\tilde{A}_{22} \in \mathbb{C}^{n_2 \times n_2}$ has all its eigenvalues inside the unit circle. Suppose also that (A, B) is stabilizable and $\kappa > 0$, $\kappa B^* M_1^* P^* P M_1 B < 2I_m$. Then the origin is globally asymptotically stable for the linear control law $u(x) = Kx$ where $K = -\kappa B^* M_1^* P^* \Lambda P M_1$.

Proof. The existence of a similarity transformation with the properties described is straightforward. Just notice that putting A into Jordan normal form is one way to accomplish this. $\tilde{A}_{11}$ is then diagonalizable because the eigenvalues on the unit circle are semisimple. Next perform a change of coordinates using M:

$$\begin{aligned}
x^+ &= Ax + Bu \\
x^+ &= M^{-1}\tilde{A}Mx + Bu \\
Mx^+ &= \tilde{A}Mx + MBu \\
\tilde{x}^+ &= \tilde{A}\tilde{x} + MBu \\
\begin{bmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{bmatrix}^+ &= \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{bmatrix} + \begin{bmatrix} M_1 Bu \\ M_2 Bu \end{bmatrix}
\end{aligned}$$

where $\tilde{x} = Mx$. Next we show that (A, B) stabilizable is equivalent to $(\tilde{A}_{11}, M_1 B)$ controllable using the Hautus lemma (Rawlings and Mayne, 2009, Lemmas 1.2 and 1.12):

$$\begin{aligned}
(A, B) \text{ stabilizable} &\iff \operatorname{rank}\begin{bmatrix}\lambda I_n - A & B\end{bmatrix} = n \quad \forall\, |\lambda| \geq 1 \\
&\iff \operatorname{rank}\begin{bmatrix}\lambda I_n - M^{-1}\tilde{A}M & B\end{bmatrix} = n \quad \forall\, |\lambda| \geq 1 \\
&\iff \operatorname{rank}\begin{bmatrix}\lambda M - \tilde{A}M & MB\end{bmatrix} = n \quad \forall\, |\lambda| \geq 1 \\
&\iff \operatorname{rank}\begin{bmatrix}(\lambda I_n - \tilde{A})M & MB\end{bmatrix} = n \quad \forall\, |\lambda| \geq 1 \\
&\iff \operatorname{rank}\begin{bmatrix}(\lambda I_n - \tilde{A})M & MB\end{bmatrix}\begin{bmatrix}M^{-1} & 0 \\ 0 & I_m\end{bmatrix} = n \quad \forall\, |\lambda| \geq 1 \\
&\iff \operatorname{rank}\begin{bmatrix}\lambda I_n - \tilde{A} & MB\end{bmatrix} = n \quad \forall\, |\lambda| \geq 1 \\
&\iff \operatorname{rank}\begin{bmatrix}\lambda I_{n_1} - \tilde{A}_{11} & 0 & M_1 B \\ 0 & \lambda I_{n_2} - \tilde{A}_{22} & M_2 B\end{bmatrix} = n \quad \forall\, |\lambda| \geq 1 \\
&\iff \operatorname{rank}\begin{bmatrix}\lambda I_{n_1} - \tilde{A}_{11} & 0 & M_1 B\end{bmatrix} = n_1 \quad \forall\, |\lambda| \geq 1 \\
&\iff \operatorname{rank}\begin{bmatrix}\lambda I_{n_1} - \tilde{A}_{11} & M_1 B\end{bmatrix} = n_1 \quad \forall\, |\lambda| \geq 1 \\
&\iff \operatorname{rank}\begin{bmatrix}\lambda I_{n_1} - \tilde{A}_{11} & M_1 B\end{bmatrix} = n_1 \quad \forall\, \lambda \in \mathbb{C} \\
&\iff (\tilde{A}_{11}, M_1 B) \text{ controllable}
\end{aligned}$$

Now we may apply Theorem 3 to the subsystem containing the $\tilde{x}^1$ modes. This gives that $u(\tilde{x}_1) = \tilde{K}\tilde{x}^1$ with $\tilde{K} = -\kappa B^* M_1^* P^* \Lambda P \tilde{x}^1$ is a stabilizing control law for $\tilde{x}^1$. That is, under closed-loop control $\tilde{x}_k^1 \to 0$. This implies that $u_k \to 0$, which implies that $\tilde{x}_k^2 \to 0$ because $\tilde{A}_{22}$ has eigenvalues inside the unit circle. $\tilde{x}$ and x are related through an invertible linear transformation, so $x_k \to 0$. In terms of the original coordinates, this control law is $u(x) = Kx$ where $K = -\kappa B^* M_1^* P^* \Lambda P M_1 x$. □

Remark. The above proof mentions using Jordan normal form to give $\tilde{A}$ the desired structure, but we state the result more generally because in practice using Jordan normal form is not numerically practicable. We have shown that a transformation with the desired properties exists, but the task of developing a numerically stable algorithm to find such a transformation is beyond the scope of this report.

Remark. It can be shown that (A, B) controllable implies $(\tilde{A}_{11}, M_1 B)$ controllable. We state Theorem 4 with the latter condition because the converse, however, is not true.

Appendix

Lemma 1. *If A is unitary, (A, B) is controllable if and only if (A^*, B) is controllable.*

Proof.

$$
\begin{aligned}
(A,B) \text{ controllable} &\iff \operatorname{rank}\big(\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}\big) = n \\
&\iff \operatorname{rank}\big((A^*)^{n-1}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}\big) = n \\
&\iff \operatorname{rank}\big(\begin{bmatrix} (A^*)^{n-1}B & (A^*)^{n-2}B & \cdots & B \end{bmatrix}\big) = n \\
&\iff \operatorname{rank}\big(\begin{bmatrix} B & A^*B & \cdots & (A^*)^{n-1}B \end{bmatrix}\big) = n \\
&\iff (A^*,B) \text{ controllable}
\end{aligned}
$$

The second equivalence follows because $(A^*)^{n-1}$ is invertible (its inverse is A^{n-1}) and the fourth equivalence follows because the two matrices have the same columns, just rearranged. □

Lemma 2. *Let (X,d) be a metric space and (x_n) a sequence with $x_n \in K \subseteq X$ where K is compact. Then the following are equivalent:*

1. *(x_n) converges.*
2. *(x_n) has exactly one subsequential limit.*

Proof. not (2) $\implies$ not (1): K is compact, so (x_n) must have at least one subsequential limit. That is, it cannot have no subsequential limits. So for this part of the proof assume that (x_n) has multiple distinct subsequential limits, say $y, z \in K$. Define $\varepsilon = d(y,z) > 0$. There exist arbitrarily large N and M such that $d(x_N, y) < \varepsilon/3$ and $d(x_M, z) < \varepsilon/3$. Now apply the triangle inequality:

$$
\begin{aligned}
d(x_N, x_M) &\geq d(y,z) - d(x_N, y) - d(x_M, z) \\
&> \varepsilon - \varepsilon/3 - \varepsilon/3 \\
&= \varepsilon/3 > 0
\end{aligned}
$$

Therefore (x_n) is not a Cauchy sequence which implies that it does not converge.

not (1) $\implies$ not (2): Assume (x_n) does not converge. K is compact, so (x_n) must have at least one subsequential limit, say, y. However $x_n \not\to y$, so there exists $\varepsilon > 0$ such that $d(x_N, y) > \varepsilon$ for arbitrarily large N. In other words, we can extract a subsequence (x_{n_k}) such that $d(x_{n_k}, y) > \varepsilon$ for all k and $n_k \to \infty$. This implies that y is not a subsequential limit of (x_{n_k}). However (x_{n_k}) must have at least one subsequential limit, say z, $z \neq y$. Notice that z is also a subsequential limit of (x_n). So (x_n) has at least two distinct subsequential limits, y and z. □

References

X. Bao, Z. Lin, and E. D. Sontag. Finite gain stabilization of discrete-time linear systems subject to actuator saturation. *Automatica*, 36(2):269–277, 2000.

J. Choi. On the stabilization of linear discrete time systems subject to input saturation. *Int. J. Sys. Sci.*, 36:241–244, 1999.

J.-S. Kim, T.-W. Yoon, A. Jadbabaie, and C. D. Persis. Input-to-state stability for neutrally stable linear systems subject to control constraints. In *Proceedings of the 43rd IEEE Conference on Decision and Control*, Atlantis, Bahamas, December 2004.

J. P. LaSalle. The stability of dynamical systems. In *Regional Conference Series in Applied Mathematics #25*. SIAM, 1976.

J. B. Rawlings and D. Q. Mayne. *Model Predictive Control: Theory and Design*. Nob Hill Publishing, Madison, WI, 2009. 668 pages, ISBN 978-0-9759377-0-9.

G. Shi, A. Saberi, and A. A. Stoorvogel. On the L_p (ℓ_p) stabilization of open-loop neutrally stable linear plants with input subject to amplitude saturation. *Int. J. Robust and Nonlinear Control*, 13:735–754, 2003.