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On the inherent robustness of optimal and suboptimal MPC

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1 Introduction

The stability of nominal model predictive control (MPC) is well characterized in the literature. Lyapunov theory forms a framework that unites both linear and nonlinear control problems, and it is amenable to a wide variety of new MPC formulations (see (Rawlings and Mayne, 2009, Ch. 2) and the references it contains). In all practical implementations, however, robustness to disturbances is a requirement. In some forms of MPC, robustness is not guaranteed; in Grimm, Messina, Tuna, and Teel (2004) it was shown that while linear quadratic MPC with polyhedral constraints is robust, there exist nominally stable implementations of nonlinear MPC that are unstable for arbitrarily small perturbations. Thus, it is vital to study the robustness of nonlinear MPC.

One method to deal with disturbances is a robust MPC formulation. In this formulation, the control problem is formulated such that, for all possible disturbances in a disturbance set known *a priori*, all constraints are satisfied. Because the robustness properties are usually guaranteed only when disturbances lie within these bounds, conservative disturbance estimates are often used. In Limón Marruedo, Álamo, and Camacho (2002), a constraint-tightening method is proposed to maintain recursive feasibility. In (Rawlings and Mayne, 2009, Ch. 3), a robust controller that optimizes over a set of control policies to recursively satisfy constraints is proposed. Min-max MPC is a robust control strategy that minimizes the controller's cost under the worst-case disturbance. However, the authors in Yu, Reble, Chen, and Allgöwer (2014) noted that min-max MPC formulations are computationally expensive to solve and often result in poor performance.

A different approach is to characterize the conditions under which nominal MPC is *inherently* robust. The authors in Grimm et al. (2004) demonstrated that continuous

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systems with continuous control laws or value functions are inherently (or what the authors call nominally) robust. More general conditions for inherent robustness are given in Grimm, Messina, Tuna, and Teel (2007). The authors in Yu et al. (2014) established that, for specific choices of terminal region and control law, optimal MPC without state constraints is inherently robust. Interestingly, the authors in Zeilinger, Morari, and Jones (2014) use soft constraints in their robust MPC formulation for additional robustness to unmodeled disturbances.

A useful definition of robustness is (local) input-to-state stability (ISS). In Limon, Alamo, Raimondo, Muñoz de la Peña, Bravo, Ferramosca, and Camacho (2009), it is established that for uniformly continuous systems and control laws, nominal MPC is ISS for systems without measurement disturbances, and in Yu et al. (2014) it is demonstrated that linear systems without state constraints (but with a terminal region) are ISS. The authors in Lazar, Heemels, and Teel (2013) establish that the existence of an ISS Lyapunov function on a robustly positive invariant set implies that the system is ISS on that set.

The above results apply to *optimal* MPC. However, in the case of nonlinear MPC, the optimal control problems are usually nonconvex NLPs; there is no guarantee that a globally (or even locally) optimal solution can be found during the sample time. Thus, the conditions for which *suboptimal* MPC is robustly stable are important for practical applications. These conditions were investigated in Lazar and Heemels (2009), where robustness with respect to state disturbances is demonstrated under the condition that the suboptimal controller is able to find a solution to within a specific degree of suboptimality. This approach was further investigated by the authors in Picasso, Desiderio, and Scattolini (2012), where they introduced auxiliary functions to take into account discontinuities and nonadditive state disturbances.

In Pannocchia, Rawlings, and Wright (2011) the authors took a different approach. They required that the suboptimal controller be initialized with a feasible initial control sequence (warm start) and it produce a control sequence no worse than the warm start. This suboptimal MPC algorithm was demonstrated to be nominally stable, and two results were established. The first result (Theorem 31) states that when their suboptimal MPC algorithm is used with both state and terminal constraints, it is robustly stable in compact sets in the interior of the feasible region. This result requires the assumption that if the system's state experiences a small perturbation, then the warm start only requires a small perturbation to become feasible. The second result states that when their algorithm is used for a system without any state constraints and the terminal constraint is replaced with an enlarged terminal penalty, the system is robustly stable. In both of these results state and measurement disturbances were considered. However, as noted by the authors of Yu et al. (2014), the assumptions used for both these results are strong enough to demonstrate that the optimal value function is continuous and therefore that optimal MPC is robust.

In this paper, we strengthen the results of Pannocchia et al. (2011). Although a control system that satisfies hard state constraints is desirable, there is no guarantee that hard state constraints can be satisfied for a disturbed system. A control system must be able to respond when state constraint satisfaction is impossible. Therefore, we soften hard state constraints using exact penalty functions, which ensure constraint satisfaction when possible. We do use a terminal control law and hard terminal region constraint. As in

Limón Marruedo et al. (2002) and Yu et al. (2014), we define the terminal region as a sublevel set of a terminal cost. In both papers, it is shown that the terminal control law brings points on the boundary of the region into the region's interior. This observation is critical for our robustness result.

Unlike Pannocchia et al. (2011), we do not assume explicitly or implicitly that the optimal value function is continuous. We include an example that has both a discontinuous optimal value function and optimal control law but nevertheless is covered by our result. This example is, to the best of the authors' knowledge, the first such example in the literature. Additionally, here we use (local) ISS as our definition of robustness. In Pannocchia et al. (2011), by contrast, there are two different definitions of robustness: RES and SRES. The latter is similar to ISS, but it has an additional constraint on the value of the system's true state. Here, robustness and ISS are formulated in terms of the measured state. These changes and the use of Proposition 19 result in clearer statements and more streamlined proofs of robust feasibility in Section 4.2.

Notation. The symbols $\mathbb{I}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$ denote the nonnegative integers and reals, respectively. The symbol $\mathbb{I}_{0:N}$ denotes the set $\{0, 1, \dots, N\}$. Given $V : X \to \mathbb{R}_{\geq 0}$ and $\tau > 0$,we define $\text{lev}_\tau V = \{x \in X \mid V(x) \leq \tau\}$. The symbol $|\cdot|$ denotes the Euclidean norm, and $\mathbb{B}$ denotes the closed ball of unit radius centered at the origin. Set deletion is denoted by $A \backslash B := \{x \in A \mid x \notin B\}$. The relation $A \subseteq B$ denotes that A is a subset of B, and the relation $A \subset B$ denotes that A is a strict subset of B. Bold symbols, e.g., $\mathbf{d}$, denote sequences and $d(k)$ denotes the element of $\mathbf{d}$ at time $k \in \mathbb{I}_{\geq 0}$. We define $\|\mathbf{d}\| := \sup_{k \geq 0} |d(k)|$. For two vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, the ordered pair $(a, b) \in \mathbb{R}^{n+m}$ represents their concatenation. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for a fixed k the function $\beta(\cdot, k)$ is of class $\mathcal{K}$ and for a fixed x the function $\beta(x, \cdot)$ is nonincreasing and satisfies $\lim_{k\to\infty} \beta(x, k) = 0$.

2 Basic definitions and assumptions

2.1 MPC problem

We consider autonomous discrete-time systems of the form

$$x^+ = f(x, u) \tag{1}$$

in which $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and x^+ denotes the successor state. Given time k, initial state x and input sequence $\mathbf{u}$, the function $\tilde{\phi}(k; x, \mathbf{u})$ denotes the open-loop state solution to the system (1). We consider here the case of hard input constraints

$$u \in \mathbb{U}, \quad x \in \mathbb{R}^n, \quad (x, u) \in \mathbb{Z} = \mathbb{R}^n \times \mathbb{U}$$

For model predictive control with a horizon of N, initial condition x and a terminal constraint, we define the set of admissible $(x, \mathbf{u})$ pairs (2), admissible inputs (3), admissible states (4) and objective function (5) by

$$\mathcal{Z}_N = \{(x, \mathbf{u}) \mid x^+ = f(x,u), (x(k), u(k)) \in \mathbb{Z} \quad \forall k \in \mathbb{I}_{0:N-1}, x(N) \in \mathbb{X}_f, x(0) = x\} \tag{2}$$

$$\mathcal{U}_N(x) = \{\mathbf{u} \mid (x, \mathbf{u}) \in \mathcal{Z}_N\} \tag{3}$$

$$\mathcal{X}_N = \{x \mid \exists \mathbf{u} \text{ such that } (x, \mathbf{u}) \in \mathcal{Z}_N\} \tag{4}$$

$$V_N(x, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N)) \tag{5}$$

The result is the optimal control problem for $x \in \mathcal{X}_N$:

$$\mathbb{P}_N(x) : V_N^0(x) = V_N(x, \mathbf{u}^0) = \min_{\mathbf{u} \in \mathcal{U}_N(x)} V_N(x, \mathbf{u}) \tag{6}$$

in which $\mathbf{u}^0$ is the optimal choice of $\mathbf{u}$. In order to establish the existence of a solution, some assumptions are needed.

Assumption 1 (Continuity of system and cost)**.** *The model $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, stage cost $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and terminal cost $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are continuous. Furthermore, for some steady state (x_s, u_s), we have that $\ell(x_s, u_s) = 0$ and $V_f(x_s) = 0$.*

For the remainder of the paper, we assume without loss of generality that $(x_s, u_s) = (0, 0)$.

Assumption 2 (Properties of constraint set)**.** *The set $\mathbb{U}$ is compact and contains the origin. The set $\mathbb{X}_f$ is defined by $\mathbb{X}_f := \operatorname{lev}_\tau V_f = \{x \in \mathbb{R}^n \mid V_f(x) \leq \tau\}$, for some $\tau > 0$.*

Assumption 3 (Stability assumption)**.** *There exists a terminal control law $\kappa_f : \mathbb{X}_f \to \mathbb{U}$ such that for all $x \in \mathbb{X}_f$*

$$f(x, \kappa_f(x)) \in \mathbb{X}_f$$
$$V_f(f(x, \kappa_f(x))) \leq V_f(x) - \ell(x, \kappa_f(x))$$

Assumption 4 (Stage cost bound)**.** *There exists a function $\alpha_\ell(\cdot) \in \mathcal{K}_\infty$ such that for all $(x, u) \in \mathbb{Z}$ we have that*

$$\ell(x, u) \geq \alpha_\ell(|(x, u)|)$$

The existence of an optimal solution can be proven from Assumptions 1 and 2 (Rawlings and Mayne, 2009, pp. 97-98). If the solution to $\mathbb{P}_N(x)$ is unique for all $x \in \mathcal{X}_N$, we can write the control law (7) and closed-loop system (8)

$$u = \kappa_N(x) := \mathbf{u}^0(0; x) \tag{7}$$

$$x^+ = f(x, \kappa_N(x)) \tag{8}$$

The case in which the solution to $\mathbb{P}_N(x)$ is not unique is considered in the following section.

Remark 5 (Assumptions)**.** *We allow the optimal steady state input to be on the boundary of* $\mathbb{U}$ *in Assumption 2. For Assumption 3, the design of the terminal controller for quadratic* $\ell(x, u)$ *is covered in (Rawlings and Mayne, 2009, pp. 136-138). Note that because* $V_f(\cdot)$ *is continuous and* $V_f(0) = 0$*, the origin lies in the interior of* $\mathbb{X}_f$*.*

Remark 6 (Comparison of assumptions in prior work)**.** *One result in Pannocchia et al. (2011) concerning robust stability, Theorem 41, uses essentially the same assumptions as the corresponding result here, Theorem 20. The only difference arises because we consider asymptotic stability rather than exponential stability; we do not require power law upper bounds for* $V_N(\cdot)$ *and* $V_f(\cdot)$ *(Assumption 4 in both papers).*

We require the following proposition on $\mathcal{K}$-functions (Rawlings and Risbeck, 2015, Proposition 14).

Proposition 7. *Let* $X \subseteq \mathbb{R}^n$ *be closed and suppose a function* $V : X \to \mathbb{R}_{\geq 0}$ *is continuous at* $x_0 \in X$ *and locally bounded on* X*. Then, there exists a function* $\alpha(\cdot) \in \mathcal{K}$ *such that for all* $x \in X$

$$|V(x) - V(x_0)| \leq \alpha(|x - x_0|)$$

From Assumption 1 and Proposition 7, we conclude that there exist functions $\alpha_2(\cdot), \alpha_f(\cdot) \in \mathcal{K}_\infty$ such that

$$V_N(x, \mathbf{u}) \leq \alpha_2(|(x, \mathbf{u})|) \qquad \forall (x, \mathbf{u}) \in \mathcal{Z}_N \tag{9}$$

$$V_f(x) \leq \alpha_f(|x|) \qquad \forall x \in \mathbb{R}^n \tag{10}$$

2.2 Suboptimal MPC

Although $\mathbb{P}_N(x)$ has a solution, for nonlinear models or constraints $\mathbb{P}_N(x)$ is nonconvex; there is no guarantee that the optimal solution can be found during the sample time. Furthermore, there may be multiple optima. Suboptimal MPC is a procedure that uses feasible solutions to $\mathbb{P}_N$ in order to design a stabilizing controller. If the optimization software is given a warm start (i.e., a feasible sequence of inputs) as an initial guess, then it can produce an improved solution or, if that fails, return the warm start. The first input is then injected and a new warm start is calculated. Given a state x and a warm start $\tilde{\mathbf{u}} \in \mathcal{U}_N(x)$, we consider the following three conditions

$$\mathbf{u} \in \mathcal{U}_N(x) \tag{11}$$

$$V_N(x, \mathbf{u}) \leq V_N(x, \tilde{\mathbf{u}}) \tag{12}$$

$$V_N(x, \mathbf{u}) \leq V_f(x) \text{ if } x \in r\mathbb{B} \tag{13}$$

in which $r > 0$ is sufficiently small so that $r\mathbb{B} \subseteq \mathbb{X}_f$. Conditions (11) and (13) define the set of feasible warm starts for all $x \in \mathcal{X}_N$:

$$\tilde{\mathcal{U}}_r(x) = \{\tilde{\mathbf{u}} \mid \tilde{\mathbf{u}} \in \mathcal{U}_N(x) \text{ and } V_N(x, \tilde{\mathbf{u}}) \leq V_f(x) \text{ if } x \in r\mathbb{B}\} \tag{14}$$

and all three conditions together define the suboptimal controller's feasible set for all $x \in \mathcal{X}_N$ and $\tilde{u} \in \tilde{\mathcal{U}}_r(x)$

$$\mathcal{U}_r(x, \tilde{\mathbf{u}}) = \{(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x),\ V_N(x, \mathbf{u}) \leq V_N(x, \tilde{\mathbf{u}}), \text{ and } V_N(x, \tilde{\mathbf{u}}) \leq V_f(x) \text{ if } x \in r\mathbb{B}\} \tag{15}$$

The suboptimal control law $\kappa_N(x, \tilde{\mathbf{u}})$ is a function of both the state x and the warm start $\tilde{\mathbf{u}}$, and may select any element of $\mathcal{U}_r(x, \tilde{\mathbf{u}})$. Note that condition (13) is a technical condition and is required so that $|x| \to 0$ implies $|\mathbf{u}| \to 0$ under the suboptimal control law.

Proposition 8. *For any $x \in \mathcal{X}_N$ and any $\tilde{\mathbf{u}} \in \mathcal{U}_N(x)$, at least one of $\{\tilde{\mathbf{u}}, \tilde{\mathbf{u}}_f(x)\}$ is a member of $\mathcal{U}_r(x)$, in which $\tilde{\mathbf{u}}_f(x) := \{\kappa_f(x), \kappa_f(f(x, \kappa_f(x))), \dots\}$.*

The proof of this proposition is provided in the appendix. Given any suboptimal input $\mathbf{u} \in \mathcal{U}_r(x, \tilde{\mathbf{u}})$, we construct the warm start for the successor state $x^+ = f(x, u(0))$ by first defining

$$\tilde{\mathbf{u}}_w(x, \mathbf{u}) := \{u(1), u(2), \dots, u(N-1), \kappa_f(\tilde{\phi}(N; x, \mathbf{u}))\}$$

and then choosing $\tilde{\mathbf{u}}^+ \in \tilde{\mathcal{U}}(x^+)$

$$\tilde{\mathbf{u}}^+ := \begin{cases} \tilde{\mathbf{u}}_f(x^+) & \text{if } x^+ \in r\mathbb{B} \text{ and} \\ & V_N(x^+, \tilde{\mathbf{u}}_f(x^+)) \leq V_N(x^+, \tilde{\mathbf{u}}_w(x, \mathbf{u})) \\ \tilde{\mathbf{u}}_w(x, \mathbf{u}) & \text{else} \end{cases} \tag{16}$$

Let $\tilde{\mathbf{u}}^+ = \zeta(x, \mathbf{u})$ denote the mapping (16). At least one of these warm starts is feasible by Proposition 8. We summarize the suboptimal algorithm as follows.

Algorithm 9 (Suboptimal MPC)**.**

- *Choose $\mathbb{X}_f$ and $V_f(\cdot)$ satisfying Assumption 3*
- *Select $r > 0$ such that $r\mathbb{B} \subseteq \mathbb{X}_f$*
- *Obtain initial state $x(0) \in \mathcal{X}_N$ and any initial warm start $\tilde{\mathbf{u}}(0) \in \tilde{\mathcal{U}}_r(x(0))$*
- *Repeat for $k = 0, 1, \dots$*
  1. *Measure current state $x(k)$*
  2. *Compute any input $\mathbf{u} \in \mathcal{U}_r(x(k), \tilde{\mathbf{u}}(k))$*
  3. *Compute the next warm start $\tilde{\mathbf{u}}(k+1)$ according to* (16)
  4. *Inject any first element of the input sequence $\mathbf{u}$ and set $k \leftarrow k+1$*

Because the control law $\kappa_N(x, \tilde{\mathbf{u}})$ is a function of the warm start, which is itself a function of the previous state and input, we next analyze the behavior of the extended state $(x, \tilde{\mathbf{u}})$.

2.3 Extended state

In Algorithm 9 we begin with a state and warm start pair and proceed from this pair to the next at the start of each time step. We denote the extended state as $z := (x, \tilde{\mathbf{u}})$. The extended state evolves according to

$$z^+ \in H(z) := \{(x^+, \tilde{\mathbf{u}}^+) \mid x^+ = f(x, u(0)), \\ \tilde{\mathbf{u}}^+ = \zeta(x, \mathbf{u}), \mathbf{u} \in \mathcal{U}_r(z)\} \tag{17}$$

in which $u(0)$ is the first element of $\mathbf{u}$. We denote by $\psi(k; z)$ any solution of (17) with initial extended state z and denote by $\phi(k; z)$ the accompanying state component of the trajectory. We also define the restriction of $\mathcal{Z}_N$ satisfying (13) by

$$\mathcal{Z}_r := \{(x, \mathbf{u}) \mid x \in \mathcal{X}_N \text{ and } \mathbf{u} \in \tilde{\mathcal{U}_r}(x)\}$$

In order to directly link the asymptotic behavior of z with that of x, the following proposition is necessary.

Proposition 10. *There exists a function $\alpha_r(\cdot) \in \mathcal{K}_\infty$ such that $|\mathbf{u}| \leq \alpha_r(|x|)$ for any $(x, \mathbf{u}) \in \mathcal{Z}_r$.*

The proof of this proposition is provided in the appendix.

3 Asymptotic stability of suboptimal MPC

3.1 Asymptotic stability of difference inclusions

As the system of interest is a difference inclusion, we present the following definitions of asymptotic stability and the associated Lyapunov functions. Consider the difference inclusion $z^+ \in H(z)$, such that $H(0) = \{0\}$.

Definition 11 (Asymptotic Stability)**.** *We say the origin of the difference inclusion $z^+ \in H(z)$ is asymptotically stable in the positive invariant set $\mathcal{Z}$ if there exists a function $\beta(\cdot) \in \mathcal{KL}$ such that for any $z \in \mathcal{Z}$ and for all $k \in \mathbb{I}_{\geq 0}$, all solutions $\psi(k; z)$ satisfy*

$$|\psi(k; z)| \leq \beta(|z|, k)$$

Definition 12 (Lyapunov Function)**.** *$V(\cdot)$ is a Lyapunov function in the positive invariant set $\mathcal{Z}$ for the difference inclusion $z^+ \in H(z)$ if there exists functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathcal{K}_\infty$ such that for all $z \in \mathcal{Z}$*

$$\alpha_1(|z|) \leq V(z) \leq \alpha_2(|z|) \tag{18}$$

$$\sup_{z^+ \in H(z)} V(z^+) \leq V(z) - \alpha_3(|z|) \tag{19}$$

Note that $V(\cdot)$ is not required to be continuous everywhere, but the existence of $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ implies that it is continuous at the origin. Because the main contribution of this work deals with robustness, we relegate the proof of asymptotic stability to the appendix.

Proposition 13. *If the set $\mathcal{Z}$ is positive invariant for the difference inclusion $z^+ \in H(z)$, $H(0) = \{0\}$, $0 \in \mathcal{Z}$, and it admits a Lyapunov function $V(\cdot)$ in $\mathcal{Z}$, then the origin is asymptotically stable in $\mathcal{Z}$.*

Theorem 14 (Nominal Asymptotic Stability)**.** *Under Assumptions 1–4, the function $V_N(z)$ is a Lyapunov function for a closed-loop system under Algorithm 9. Therefore the system is asymptotically stable.*

4 Robust asymptotic stability of suboptimal MPC

4.1 Disturbances and robust stability definitions

For robustness analysis, we consider the following perturbed system

$$x^+ = f(x, u) + d \tag{20}$$
$$u = \kappa_N(x + e, \tilde{\mathbf{u}}) \tag{21}$$

where d is the additive process disturbance and e is the additive measurement disturbance. In order to simplify analysis, we reformulate this equation in terms of the measured state $x_m := x + e$.

$$\begin{aligned} x_m^+ &= f(x_m - e, u) + d + e^+ \\ u &= \kappa_N(x_m, \tilde{\mathbf{u}}) \end{aligned} \tag{22}$$

in which e^+ is the measurement disturbance at the next time step. We replace conditions (11)–(13) with the following for all $x_m \in \mathcal{X}_N$:

$$\mathbf{u} \in \mathcal{U}_N(x_m) \tag{23}$$
$$V_N(x_m, \mathbf{u}) \leq V_N(x_m, \tilde{\mathbf{u}}) \tag{24}$$
$$V_N(x_m, \mathbf{u}) \leq V_f(x_m) \text{ if } x_m \in r\mathbb{B} \tag{25}$$

and the controller's resulting feasible set is $\mathcal{U}_r(x_m, \tilde{\mathbf{u}})$. As before, the suboptimal control law can select any member of this set, $\kappa_N(x_m, \tilde{\mathbf{u}}) \in \mathcal{U}_r(x_m, \tilde{\mathbf{u}})$.

Algorithm 9 is used by replacing the true state x with the measured state x_m. Therefore, we define the measured extended state by

$$z_m := (x_m, \tilde{\mathbf{u}})$$

The procedure to generate the next warm start is also based on the measured state

$$\tilde{\mathbf{u}}^+ = \zeta(x_m, \mathbf{u}) \tag{26}$$

The perturbed extended system then evolves as

$$\begin{aligned} z_m^+ \in H_{ed}(z_m) := \{(x_m^+, \tilde{\mathbf{u}}^+) \mid \\ x_m^+ &= f(x_m - e, u(0)) + d + e^+, \\ \tilde{\mathbf{u}}^+ &= \zeta(x_m, \mathbf{u}), \mathbf{u} \in \mathcal{U}_r(z_m)\} \end{aligned} \tag{27}$$

Finally, for given disturbance and measurement error sequences $\mathbf{d}$ and $\mathbf{e}$ and initial extended state $z_m = (x_m, \tilde{\mathbf{u}})$ we denote an arbitrary solution of (27) at time k by $\psi_{ed}(k; z_m)$ and the corresponding state by $\phi_{ed}(k; z_m)$.

Definition 15 (Robust Positive Invariance)**.** *A set $\mathcal{S} \subseteq \mathbb{R}^n$ is robustly positive invariant with respect to some difference inclusion $z^+ \in H_w(z)$ if for all $z \in \mathcal{S}$ and for all disturbances w satisfying $\|w\| \leq \delta$ for some $\delta > 0$, we have that $H_w(z) \subseteq \mathcal{S}$.*

Next, we define robust asymptotic stability as input-to-state stability on a robustly positive invariant set.

Definition 16 (Robust Asymptotic Stability)**.** *The origin of a perturbed difference inclusion $z^+ \in H_w(z_m)$ is robustly asymptotically stable in $\mathcal{S}$ if, for all disturbance sequences $\mathbf{w}$ satisfying $\|\mathbf{w}\| \leq \delta$ with $\delta > 0$, $\mathcal{S}$ is robustly positive invariant and there exists $\beta(\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$ such that for each $z \in \mathcal{S}$ and for all $z^+ \in H_w(z)$, we have for all $k \in \mathbb{I}_{\geq 0}$ that*

$$|\psi_w(k; z, \tilde{\mathbf{u}})| \leq \beta(|z|, k) + \gamma(\|\mathbf{w}\|) \tag{28}$$

In our case, we define $w(k) := (d(k), e(k), e(k+1))$. In order to demonstrate ISS, we define an ISS Lyapunov function for a difference inclusion, similar to ISS Lyapunov function defined in Jiang and Wang (2001) and Lazar et al. (2013).

Definition 17 (ISS Lyapunov Function)**.** *$V(\cdot)$ is an ISS Lyapunov function in the positive invariant set $\mathcal{Z}$ for the difference inclusion $z^+ \in H_w(z)$ if there exists functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathcal{K}_\infty$ and function $\sigma(\cdot) \in \mathcal{K}$ such that for all $z \in \mathcal{Z}$ and $\|\mathbf{w}\| \leq \delta$*

$$\alpha_1(|z|) \leq V(z) \leq \alpha_2(|z|) \tag{29}$$

$$\sup_{z^+ \in H_w(z)} V(z^+) \leq V(z) - \alpha_3(|z|) + \sigma(\|\mathbf{w}\|) \tag{30}$$

Proposition 18. *If a difference inclusion $z^+ \in H(z, w)$ admits an ISS Lyapunov function in a positive invariant set $\mathcal{Z}$ for all $\|\mathbf{w}\| \leq \delta$ for some $\delta > 0$, then the origin is robustly asymptotically stable for all $\|\mathbf{w}\| \leq \delta$.*

The proof is similar to that provided in Jiang and Wang (2001) and is provided in the appendix.

We require a result that allows us, for $f : \mathcal{D} \to \mathbb{R}^n$ and $x, y \in \mathcal{D}$, to bound $|f(x) - f(y)|$ in terms of $|x - y|$. This bound can be obtained if $f(\cdot)$ is continuous and $\mathcal{D}$ is compact; by the Heine-Cantor Theorem $f(\cdot)$ is uniformly continuous on $\mathcal{D}$ and, by a result in Rawlings and Risbeck (2015) there exists some $\sigma(\cdot) \in \mathcal{K}$ such that $|f(x) - f(y)| \leq \sigma(|x - y|)$. However, $\mathcal{D}$ may not be compact. Proposition 19 permits the point y to be anywhere in $\mathcal{D}$ so long as x remains in some compact set $\mathcal{C}$.

Proposition 19. *Let $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$, with $\mathcal{C}$ compact, $\mathcal{D}$ closed, and $f : \mathcal{D} \to \mathbb{R}^n$ continuous. Then there exists $\sigma(\cdot) \in \mathcal{K}_\infty$ such that for all $x \in \mathcal{C}$ and $y \in \mathcal{D}$, we have that $|f(x) - f(y)| \leq \sigma(|x - y|)$.*

This proposition is necessary to bound the distance between the nominal successor state and the measured successor state; if we used uniform continuity on some set $\mathcal{C}$ to bound that distance, we could not use that bound to conclude that $\mathcal{C}$ is robustly positive invariant. The proof of Proposition 19 is provided in the appendix.

4.2 Main results

Robust asymptotic stability of (27) is demonstrated by a three part argument. First, robust feasibility is demonstrated, showing that a new input resulting in a sufficient cost decrease can be calculated using $\kappa_f(\cdot)$. Second, it is demonstrated that there exists a positive invariant compact set $\mathcal{S}$ containing the origin. Finally, an ISS Lyapunov function valid inside $\mathcal{S}$ is found, which by Proposition 18 implies the origin is robustly asymptotically stable.

Theorem 20. *Under Assumptions 1–4, for every $\rho > 0$ there exists some $\delta > 0$ such that if $\|w\| \leq \delta$ the disturbed extended system* (20) *is robustly asymptotically stable on the set $\mathcal{S} := \operatorname{lev}_\rho V_N \cap \mathcal{Z}_N$. In terms of x_m, if Algorithm 9 is initialized with a warm start $\tilde{\mathbf{u}}$ such that $(x_m, \tilde{u}) \in \mathcal{S}$, then the set $\mathcal{C} := \{x \mid \exists \tilde{\mathbf{u}} \text{ such that } (x, \tilde{\mathbf{u}}) \in \mathcal{S}\}$ is robustly positive invariant and there exist $\beta(\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$ such that $|\phi_{ed}(k; x_m, \tilde{\mathbf{u}})| \leq \beta(|x_m|, k) + \gamma(\|\mathbf{w}\|)$.*

Proof. Part I: Robust Feasibility

Note that $\mathcal{S}$ is compact because $\mathcal{Z}_N$ is closed and $\operatorname{lev}_\rho V_N$ is compact for all finite ρ. Also note that $x_m \in \mathcal{X}_N$, but x is not necessarily in $\mathcal{X}_N$. By Assumption 1 we have that $f(\cdot)$ is continuous. Then, by applying Proposition 19 to (27), for some $\tilde{\sigma}_x(\cdot) \in \mathcal{K}_\infty$ we have that

$$|f(x_m - e, u(0)) - \tilde{x}| \leq \tilde{\sigma}_x(|x_m - e - x_m|) = \tilde{\sigma}_x(|e|)$$

with $u(0) = \kappa_N(x_m, \tilde{\mathbf{u}})$ and note that $f(x_m, u(0)) := \tilde{x}^+$. We also have that

$$\left|x^+ - f(x_m - e, u(0))\right| \leq |d| \quad \text{and} \quad \left|x_m^+ - x^+\right| \leq \left|e^+\right|$$

Combining the above bounds, we have that

$$\begin{aligned}
\left|x_m^+ - \tilde{x}^+\right| &\leq \left|f(x_m - e, u(0)) - \tilde{x}^+\right| \\
&\quad + \left|x^+ - f(x_m - e, u(0))\right| + \left|x_m^+ - x^+\right| \\
&\leq \tilde{\sigma}_x(|e|) + |d| + \left|e^+\right| \leq \tilde{\sigma}_x(|w|) + |w| + |w|
\end{aligned}$$

and recall that $w := (d, e, e^+)$, so the last step follows because $|a| \leq |(a, b)|$, in which a and b are both vectors. We define $\tilde{\sigma}_x(|w|) + 2\,|w| := \sigma_x(|w|)$ and note that $\sigma_x(\cdot) \in \mathcal{K}_\infty$.

Figure 1 illustrates the sets and trajectories used in this argument. Recall that by Assumption 1, both $f(\cdot)$ and $V_N(\cdot)$ are continuous. Therefore $\tilde{\phi}(N; x, \mathbf{u})$ is continuous in x as it is defined as a finite number of compositions of $f(\cdot)$. Similarly, $V_f(\tilde{\phi}(N; x, \mathbf{u}))$ is continuous in x. From the previous control sequence $\mathbf{u}$ satisfying (23)–(25), define the next warm start $\tilde{\mathbf{u}}^+$ using (26). Let $x_m(N) := \tilde{\phi}(N; x_m, \tilde{\mathbf{u}})$, $\tilde{x}^+(N) := \tilde{\phi}(N; \tilde{x}^+, \tilde{\mathbf{u}}^+)$, and $x_m^+(N) := \tilde{\phi}(N; x_m^+, \tilde{\mathbf{u}}^+)$. Therefore, by Proposition 19, we have that

$$\begin{aligned}
\left|V_f(x_m^+(N))) - V_f(\tilde{x}^+(N))\right| &\leq \tilde{\sigma}_f(\left|x_m^+ - \tilde{x}\right|) \\
&\leq \tilde{\sigma}_f(\sigma_x(|w|)) := \sigma_f(|w|)
\end{aligned}$$

and note that because $\sigma_x(\cdot), \tilde{\sigma}_f(\cdot) \in \mathcal{K}_\infty$, we have that $\sigma_f(\cdot) \in \mathcal{K}_\infty$. Regardless of whether $V_f(x_m^+(N)) - V_f(\tilde{x}^+(N))$ is positive or negative we have that

$$V_f(x_m^+(N)) \leq V_f(\tilde{x}^+(N)) + \sigma_f(|w|) \tag{31}$$

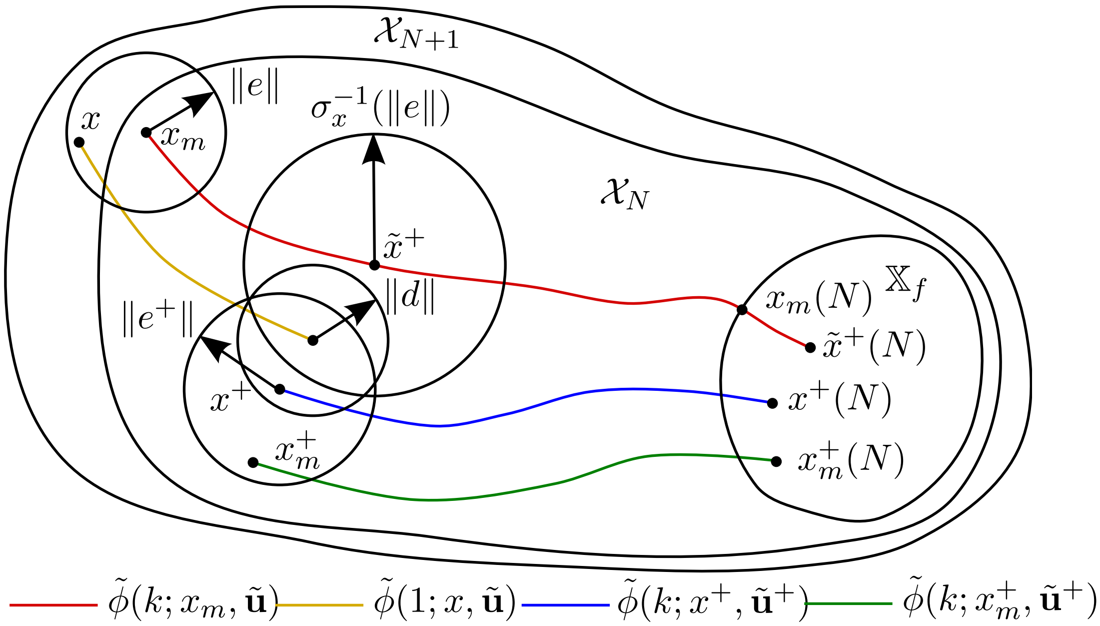

Figure 1: Though the nominal trajectory may terminate on the boundary of $\mathbb{X}_f$, the perturbed trajectories terminate in $\mathbb{X}_f$ nevertheless. Note that $x^+(N) := \tilde{\phi}(N; x^+, \tilde{\mathbf{u}}^+)$.

Since $x_m(N) \in \mathbb{X}_f$, by Assumption 3 we have that

$$\begin{aligned}
V_f(\tilde{x}^+(N)) - V_f(x_m(N)) &\leq -\ell(x_m(N), \kappa_f(x_m(N))) \\
&\leq -\alpha_\ell(|(x_m(N), \kappa_f(x_m(N)))|) \\
&\leq -\alpha_\ell(|x_m(N)|)
\end{aligned}$$

Recall that $\mathbb{X}_f := \text{lev}_\tau V_f$. To demonstrate that $x_m^+(N) \in \mathbb{X}_f$, we define an inner sublevel set $\mathcal{X}_f := \text{lev}_{\tau/2} V_f$. We then bound $|w|$ such that if $\tilde{x}^+(N) \in \mathbb{X}_f \backslash \mathcal{X}_f$ the decrease in cost shown above is large enough to compensate for the perturbation, and therefore $x_m^+(N) \in \mathbb{X}_f$. On the other hand if $\tilde{x}^+(N) \in \mathcal{X}_f$, we bound $|w|$ such that cost cannot increase enough to leave $\mathbb{X}_f$ in a single step, and therefore cannot avoid the region where cost decreases.

Case I: $V_f(x_m(N)) \geq \tau/2$.
By (10), we have that $|x_m(N)| \geq \alpha_f^{-1}(\tau/2)$. Therefore, we have that $V_f(\tilde{x}^+(N)) - V_f(x_m(N)) \leq -\alpha_\ell(\alpha_f^{-1}(\tau/2))$ which implies $V_f(\tilde{x}^+(N)) \leq \tau - \alpha_\ell(\alpha_f^{-1}(\tau/2))$.

Case II: $V_f(x_m(N)) < \tau/2$.
Since $V_f(\tilde{x}^+(N)) - V_f(x_m(N)) \leq 0$ we immediately have that $V_f(\tilde{x}^+(N)) \leq \tau/2$.

We conclude that

$$V_f(\tilde{x}^+(N)) \leq \tau - \gamma$$

in which $\gamma = \min\{\tau/2, \alpha_\ell(\alpha_f^{-1}(\tau/2))\}$ and we note that γ is independent of (z_m, e, d, e^+).

Finally, by combining (31) with the above, we have that :

$$V_f(x_m^+(N)) \leq V_f(\tilde{x}^+(N)) + \sigma_f(|w|) \leq \tau - \gamma + \sigma_f(|w|)$$

and by bounding $|w| \leq \sigma_f^{-1}(\gamma) := \delta_1$, we have that $V_f(x_m^+(N)) \leq \tau$ and therefore $x_m^+(N) \in \mathbb{X}_f$. Thus, the measured trajectory is feasible without any optimization. Note that this result applies only for $z_m \in \mathcal{S}$. In order to complete the proof of recursive feasibility, we must have $z_m^+ \in \mathcal{S}$, which is established next.

Part II: Robust Positive Invariance

Recall that $\mathcal{S} := \text{lev}_\rho V_N \cap \mathcal{Z}_N$. Here we demonstrate that if $V_N(z_m) \leq \rho$ (and thereby $z_m \in \mathcal{S}$), then $V_N(z_m^+) \leq \rho$, and thus $z_m^+ \in \mathcal{S}$. As before, we divide $\mathcal{S}$ into inner and outer regions. In the outer region, cost must decrease, while in the inner region cost cannot increase enough to leave $\mathcal{S}$.

Since $z_m^+ := (x_m^+, \tilde{u}^+)$ and $\tilde{z}^+ := (\tilde{x}^+, \tilde{u}^+)$, we have that $|z_m^+ - \tilde{z}^+| = |x_m^+ - \tilde{x}^+| \leq \sigma_x(|w|)$. Since $V_N(\cdot)$ is continuous, we can apply Proposition 19 to obtain

$$\begin{aligned} \left|V(z_m^+) - V(\tilde{z}^+)\right| &\leq \tilde{\sigma}_\rho(\left|z_m^+ - \tilde{z}^+\right|) \\ &\leq \tilde{\sigma}_\rho(\sigma_x(|w|)) := \sigma_\rho(|w|) \end{aligned}$$

As before, we drop the absolute value on the left and rearrange to obtain

$$V(z_m^+) \leq V(\tilde{z}^+) + \sigma_\rho(|w|)$$

and because $V_N(\cdot)$ is a Lyapunov function for the nominal system, we have that

$$V_N(z_m^+) \leq V_N(\tilde{z}^+) + \sigma_\rho(|w|) \leq V_N(z_m) - \alpha_3(|z_m|) + \sigma_\rho(|w|)$$

Case I: $V_N(z_m) \geq \rho/2$. Note that because $\rho/2 \leq V_N(z_m) \leq \alpha_2(|z_m|)$, we know that $\alpha_2^{-1}(\rho/2) \leq |z_m|$. Furthermore, because $V_N(z_m) \leq \rho$,

$$\begin{aligned} V_N(z_m^+) &\leq V_N(z_m) - \alpha_3(|z_m|) + \sigma_\rho(|w|) \\ &\leq \rho - \alpha_3(\alpha_2^{-1}(\rho/2)) + \sigma_\rho(|w|) \end{aligned}$$

In order to conclude that $V_N(z_m^+) \leq \rho$, we require $-\alpha_3(\alpha_2^{-1}(\rho/2)) + \sigma_\rho(|w|) \leq 0$ and therefore choosing $\delta_2 := \sigma_\rho^{-1}(\alpha_3(\alpha_2^{-1}(\rho/2))) > 0$, we have that $V_N(z_m^+) \leq \rho$ for every $|w| \leq \delta_2$ for z_m satisfying Case I.

Case II: $V_N(z_m) < \rho/2$. Now we have that

$$V_N(z_m^+) \leq V_N(z_m) - \alpha_3(|z_m|) + \sigma_\rho(|w|) < \rho/2 + \sigma_\rho(|w|)$$

because in the worst case $|z_m| = 0$, and no cost decrease occurs. Nevertheless, if $|w| \leq \sigma_\rho^{-1}(\rho/2) := \delta_3$, we have that $V_N(z_m^+) \leq \rho$ and thus $z_m^+ \in \mathcal{S}$.

Finally, note that since $z_m^+ \in \mathcal{S}$ we have recursive feasibility. Therefore, we conclude that because $z_m \in \mathcal{S}$ and $V_N(z_m) \leq \rho$, by induction, $\mathcal{S}$ is positive invariant under control law (26) if $|(d, e, e^+)| \leq \delta := \min\{\delta_1, \delta_2, \delta_3\}$.

Remark 21. *It may seem that the system becomes more robust to disturbances as ρ increases, because both δ_2 and δ_3 are equal to some $\mathcal{K}$ functions of ρ. These $\mathcal{K}$ functions seem to imply that if ρ is large enough, then δ_1 is the only limit present on the magnitude of w required for robustness. Recall that $\sigma_\rho(\cdot) \in \mathcal{K}_\infty$ was generated by invoking Proposition 19 on $\mathcal{S}$. Therefore, if we have $\rho_1 < \rho_2$, then $\mathcal{S}_{\rho_1} \subset \mathcal{S}_{\rho_2}$. As a result $\sigma_{\rho_1}(s) \leq \sigma_{\rho_2}(s)$ for all $s \in \mathbb{R}_{\geq 0}$, because if we have $z_1, z_2 \in \mathcal{S}_{\rho_1}$, then $z_1, z_2 \in \mathcal{S}_{\rho_2}$ as well; the requirement $|V(z_1) - V(z_2)| \leq \sigma_{\rho_2}(|z_1 - z_2|)$ must be at least as restrictive as $|V(z_1) - V(z_2)| \leq \sigma_{\rho_1}(|z_1 - z_2|)$. Therefore, increasing the size of ρ has an ambiguous effect on the size of δ, which may decrease as ρ increases.*

Part III: Robust Asymptotic Stability

We have that

$$\begin{aligned} \alpha_1(|z_m|) &\leq V_N(z_m) \leq \alpha_2(|z_m|) \\ V_N(z_m^+) &\leq V_N(z_m) - \alpha_3(|z_m|) + \sigma_\rho(|w|) \end{aligned}$$

for all $z_m \in \mathcal{S}$, which is positive invariant for $|w| \leq \delta$. Therefore $V_N(z_m)$ is an ISS-Lyapunov function in $\mathcal{S}$. Thus by Proposition 18, the perturbed system (23) controlled by Algorithm 9 is ISS and therefore robustly asymptotically stable according to Definition 16. Thus, there exists $\beta(\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$ such that $|\psi_{ed}(k; z)| \leq \beta(|z|, k) + \gamma(\|w\|)$. Finally, from Proposition 10 we have $\alpha_r(\cdot) \in \mathcal{K}_\infty$ such that $|u| \leq \alpha_r(|x|)$. Then we have $|z| \leq |x| + \alpha_r(|x|) := \alpha_{r*}(|x|)$. Therefore, we have that $\beta(|z|, k) \leq \beta(\alpha_{r*}(|x|), k) := \beta_x(|x|, k)$. Because $|x| \leq |z|$, we have that $\phi(k; x, \tilde{\mathbf{u}}) \leq \beta_x(|x|, k) + \gamma(\|w\|)$. □

5 Example: Discontinuous Optimal Cost

We conclude with an example of a system for which the optimal cost function is discontinuous but is nevertheless robustly stable. Consider the following discrete-time system:

$$x^+ = x\left(5u + \frac{1}{1 + (x-1)^2}\right) \tag{32}$$

with $u \in \mathbb{U} = [0, 1]$. Choosing the terminal set $\mathbb{X}_f = [-0.5, 0.5]$, on which the system

$$x^+ = f(x, \kappa_f(x)) := f_{\kappa_f}(x)$$

is asymptotically stable under the control law $\kappa_f(x) = 0$. As a stage cost, we choose

$$\ell(x, u) = x^2 + 10u \tag{33}$$

which greatly penalizes use of the control input u. In particular, this cost satisfies the condition

$$\begin{aligned} \ell(x, u) &\geq \alpha_\ell(|(x, u)|) \\ \alpha_\ell(s) &:= \frac{1}{2}s^2 \end{aligned}$$

for all $(x, u) \in \mathbb{R} \times [0, 1]$. Recalling that the terminal cost must satisfy

$$V_f(f_{\kappa_f}(x))) \leq V_f(x) - \ell(x, \kappa_f(x))$$

we note that a suitable terminal cost is given by

$$V_f(x) := 10x^2$$

Consider first the case of $N = 1$. We require $u \in [0, 1]$ that $f(x, u) \in [-0.5, 0.5]$. Noting that $|f(x, u)| \geq |f(x, 0)|$ for any $u \in [0, 1]$, the feasible set $\mathcal{X}_1$ is given by the points where $u = 0$ moves x into $\mathbb{X}_f$, i.e.,

$$-0.5 \leq \frac{x}{1 + (x - 1)^2} \leq 0.5$$

which yields $\mathcal{X}_1 = (-\infty, a_1] \cup [b_1, \infty)$, with $a_1 = 2 - \sqrt{2}$ and $b_1 = 2 + \sqrt{2}$.

Consider now the case of $N = 2$. Due to the terminal constraint, we require that $f(x, u) \in \mathcal{X}_1$. First, we check where $u = 0$ is feasible. We require

$$\frac{x}{1 + (x - 1)^2} \in (-\infty, a_1] \cup [b_1, \infty)$$

which is satisfied for $x \in (-\infty, a_2] \cup [b_2, \infty)$ with

$$a_2, b_2 = \frac{2a_1 + 1 \pm \sqrt{1 + 4a_1 - 4a_1^2}}{2a_1}$$

For $x \in (a_2, b_2)$, the condition becomes $f(x, u) \geq b_1$. Thus, a feasible control input is given by

$$u_2^0(x) = \begin{cases} \frac{1}{5}\left(\frac{b_1}{x} - \frac{1}{1 + (x - 1)^2}\right) & x \in (a_2, b_2) \\ 0 & \text{else} \end{cases}$$

We establish that this control law is optimal in the appendix. The corresponding optimal cost is shown in Figure 2.

Because robustness is achieved via feasibility of the warm start, we demonstrate this property in Figure 3. Although the terminal state x_2 does not always satisfy the terminal constraint due to the disturbances, after additional application of κ_f to arrive at x_3, the terminal constraint is satisfied. Thus, the warm-start is feasible.

The argument for robust feasibility implies that while $x_m \in \mathcal{X}_N$, x need not be in $\mathcal{X}_N$. Because we have that $\tilde{\phi}(N; x_m^+, \tilde{\mathbf{u}}^+) \in \mathbb{X}_f$ so long as $|w| \leq \delta_1$, we also have that $\tilde{\phi}(N; x^+, \tilde{\mathbf{u}}^+) \in \mathbb{X}_f$ because e^+ does not disturb that trajectory. Therefore, we have constructed a sequence of control inputs which controls x into $\mathbb{X}_f$ in $N + 1$ timesteps, and therefore $x \in \mathcal{X}_{N+1}$. Furthermore, this implies the set of x which can be controlled into $\mathbb{X}_f$ in any number of timesteps must be open. This fact is established in Corollary 22.

Corollary 22. *The set* $\mathcal{X}_\infty := \bigcup_{k=1}^{\infty} \mathcal{X}_k$ *is open.*

Proof. A set is open if it contains a neighborhood for each of its elements. Take $x \in \mathcal{X}_N \subseteq \mathcal{X}_\infty$ for some N. As we have shown above, there exists $\delta_1 > 0$ such that if $|e| \leq \delta_1, x \in \mathcal{X}_N$ implies $x + e \in \mathcal{X}_{N+1} \subseteq \mathcal{X}_\infty$. So every point $x \in \mathcal{X}_\infty$ has some neighborhood also in $\mathcal{X}_\infty$. Thus, $\mathcal{X}_\infty$ is open. $\square$

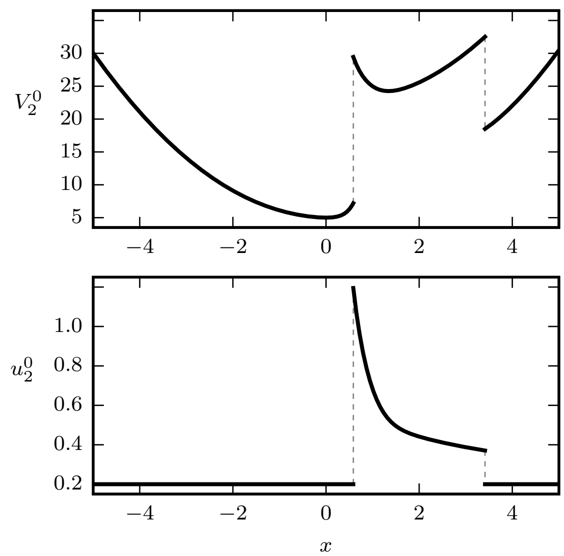

Figure 2: Optimal cost and control input for $N = 2$. Dashed gray lines show discontinuities (both curves are lower-semicontinuous).

5.1 Analysis

First, we contrast the robust stability result of Pannocchia et al. (2011), Theorem 41, which most resembles the result here, Theorem 20. Both Theorem 41 and Theorem 20 do not use state constraints. Theorem 41 does not explicitly enforce the terminal constraint, but increases the terminal cost such that the terminal constraint is fulfilled for a certain set of initial conditions. Theorem 20, by contrast, does not require the terminal constraint or terminal penalty to be altered to ensure robustness. Theorem 41 establishes robustness for compact sets on the interior of the set $\mathbb{X}_0$ (analogous to $\mathcal{C}$ here) while Theorem 20 establishes robustness for all of $\mathcal{C}$. Finally, in order to enlarge $\mathbb{X}_0$, the terminal penalty must also increase. Theorem 20 applies to arbitrarily large $\mathcal{C}$ without altering Algorithm 9.

The set $\mathcal{S} = \text{lev}_\rho V_N \cap \mathcal{Z}_N$ facilitates many of the tools necessary to establish Theorem 20. Because it is compact, Proposition 19 can be used to provide $\mathcal{K}$ functions that capture the continuity properties of $f(\cdot)$, $\ell(\cdot)$, and $V_f(\cdot)$. Because it is a sublevel set of the Lyapunov function $V_N(\cdot)$, it can be established to be positive invariant using the contractivity of $V_N(\cdot)$. However, note that while $\mathcal{C}$, the projection of $\mathcal{S}$ onto $\mathbb{R}^n$, can encompass all of $\mathcal{X}_N$ so long as it is bounded, it cannot encompass all of $\mathbb{R}^n$. This limitation exists because, as observed in Remark 21, δ may decrease as ρ increases; in the limit as ρ goes to infinity, δ may shrink to zero. If the functions $f(\cdot)$, $\ell(\cdot)$, and $V_f(\cdot)$ are all uniformly continuous on $\mathbb{R}^n$, then inherent robustness can be established on all of $\mathbb{R}^n$.

The specific form of $\mathbb{X}_f$ as a sublevel set of $V_f(\cdot)$ provides the rest of the tools necessary to establish Theorem 20. The contractivity of the terminal control law ensures that the terminal constraint is robustly satisfied at the $N+1$ step. This allows space for disturbances

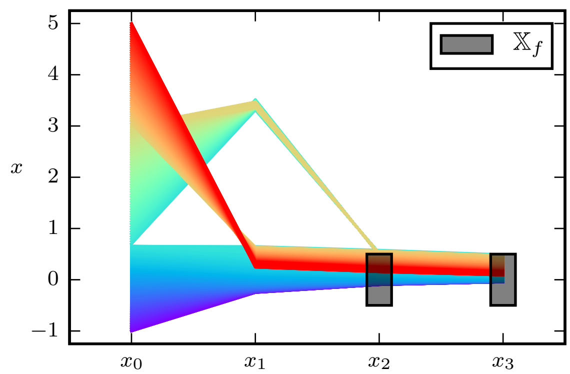

Figure 3: Warm start with $|d| \leq 0.05$ and $|e| \leq 0.01$. Evolution is $x_1 = f(x_0 - e, \kappa_2(x_0)) + d$ followed by $x_{k+1} = f(x_k, 0)$.

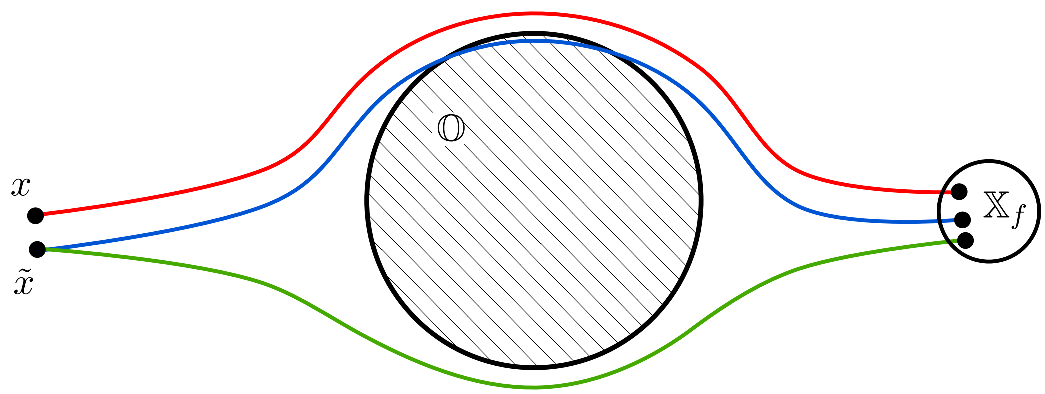

Figure 4: If x is perturbed to $\tilde{x}$, there is no guarantee that its trajectory still avoids the obstacle set $\mathbb{O}$. If small violations of state constraints are allowed, however, the trajectory remains feasible.

without the warm start $\tilde{\mathbf{u}}^+$ becoming infeasible. However, in the usual case of $V_f(x) = x'Px$ for some positive definite matrix $P \in \mathbb{R}^{n\times n}$, the terminal region is an ellipsoid. It is often beneficial to formulate the optimal control problem with only polyhedral constraints. An approximation $\tilde{\mathbb{X}}_f \subset \text{lev}_\tau V_f$ is sufficient for robustness so long as $\text{lev}_{\tau-\gamma} V_f \subset \text{interior}(\tilde{\mathbb{X}}_f)$, in which $\gamma = \min\{\tau/2, \alpha_\ell(\alpha_f^{-1}(\tau/2))\}$. Note that in this approximation, the minimum distance between $\text{lev}_{\tau-\gamma} V_f$ and $\tilde{\mathbb{X}}_f$ represents the degree of robustness.

The softening of hard state constraints is essential to this analysis. There is no guarantee that arbitrary disturbances respect them, as illustrated in Figure 4. If they are violated, then the solver must restore feasibility at a given timestep. If it does not, then no input is generated and a failsafe routine must take over. When the constraints are softened, the control law is still defined even if the solver fails to generate a feasible sequence at one timestep. Feasibility may then be restored over the next several timesteps. If the constraints are critical and even small violations cannot be tolerated then the failsafe routine must be activated in both cases— not as a result of the softened state constraints, but because either there exists no feasible solution or there is not enough time for the solver to find it. If the constraints in question are the result of physical limitations of the system, such as not permitting negative volumes or concentration, then the system evolution enforces them even if the controller does not.

In the presence of soft constraints, the optimal control law may result in small constraint violations. If this is undesirable, it can be eliminated by tightening the soft constraints. By applying a sufficiently large penalty, it can be ensured that all control sequences which satisfy the original state constraints have a cost less than those which do not. Thus constraint violations can be removed from the optimal control law.

6 Conclusions

We have established that suboptimal MPC, satisfying mild assumptions, is both stabilizing and inherently robust. Furthermore, we have illustrated that these assumptions are weak enough to permit discontinuous optimal value functions. In general, optimal control problems with nonlinear system dynamics are nonconvex NLPs, and there is no guarantee that these problems can be solved to optimality during the sample time. However, a feasible warm start is sufficient for our algorithm. In practice, optimization software is expected to generate an input much better than the warm start. Any extra cost reduction achieved by this input imparts the suboptimal controller with additional robustness. If an algorithm cannot find even a local solution during the sampling time, it is necessary for it to produce iterates that satisfy the input, state evolution, and terminal constraints. One such algorithm is proposed in Tenny, Wright, and Rawlings (2004), but the creation of further algorithms with these properties is an active area of research.

This work can be extended in a number of ways. First, note that we assumed only that the set of feasible inputs is compact; we did not assume that the steady-state input lies in its interior. Therefore, Assumption 1 permits sets of feasible inputs that do not have interiors, i.e., those that describe *discrete* inputs. Therefore, our result can be extended to MPC with discrete actuators. Second, as observed in (Rawlings and Mayne, 2009, Ch. 6), distributed MPC can be analyzed through the lens of suboptimal MPC; the techniques used in this paper may be used to analyze the inherent robustness of distributed MPC. Finally, we have analyzed systems that have positive definite stage costs. Economic MPC deals with systems that may have indefinite stage costs. Thus future work may be directed to analyze the inherent robustness of economic MPC.

Acknowledgments

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7 Appendix

Here we present the proofs omitted from the main body of the paper. Note that some of these proofs require additional propositions that are not mentioned in the body of the paper.

7.1 Suboptimal MPC

Proposition 8. *For any $x \in \mathcal{X}_N$ and any $\tilde{\mathbf{u}} \in \mathcal{U}_N(x)$, at least one of $\{\tilde{\mathbf{u}}, \tilde{\mathbf{u}}_f(x)\}$ is a member of $\mathcal{U}_r(x)$, in which $\tilde{\mathbf{u}}_f(x) := \{\kappa_f(x), \kappa_f(f(x, \kappa_f(x))), \dots\}$.*

Proof. First, the existence of an optimal solution to (6) can be proven from Assumptions 1 and 2 (Rawlings and Mayne, 2009, pp. 97-98). Next, we note that any optimal solution $\mathbf{u}^0(x)$ satisfies (11) and (12). For $x \in r\mathbb{B}$, we consider the following alternate warm start

$$\tilde{\mathbf{u}}_f(x) := \{\kappa_f(x), \kappa_f(f(x, \kappa_f(x))), \dots\} \tag{34}$$

Applying Assumption 3 for $k = 0, \dots, N-1$, we have that

$$V_N(x, \tilde{\mathbf{u}}_f(x)) = \sum_{k=0}^{N-1} \ell(x(k), u_f(k)) + V_f(x(N)) \leq V_f(x)$$

Therefore, $\tilde{\mathbf{u}}_f(x)$ satisfies (13) and therefore $\tilde{\mathbf{u}}_f(x) \in \mathcal{U}(x, \tilde{\mathbf{u}})$. Since $\tilde{\mathbf{u}}_f \in \mathcal{U}_N(x)$ and by optimality $V_N(x, \mathbf{u}^0(x)) \leq V_N(x, \tilde{\mathbf{u}}_f)$, we have that $\mathbf{u}^0(x) \in \mathcal{U}_r(x, \tilde{\mathbf{u}})$. If $x \notin r\mathbb{B}$, then (13) does not apply and $\tilde{\mathbf{u}} \in \mathcal{U}_r(x, \tilde{\mathbf{u}})$. □

7.2 Asymptotic stability of suboptimal MPC

Proposition 23. *For all $\tilde{u} \in \tilde{\mathcal{U}}_r(0)$, $\kappa_N(0, \tilde{u}) = \{0\}$.*

Proof. By Assumption 1, we have that $V_f(\cdot)$ is non-negative. Therefore, by Assumption 4, we have that

$$V_N(x, \mathbf{u}) \geq \sum_{k=0}^{N-1} \alpha_\ell(|(x(k), u(k))|)$$

By (1) in Rawlings and Ji (2012), we have that

$$\sum_{k=0}^{N-1} \alpha_\ell(|(x(k), u(k))|) \geq \alpha_\ell\left(\frac{1}{N}\sum_{k=0}^{N-1} |(x(k), u(k))|\right)$$

Next, for the Euclidean norm we have that $|a| + |b| = |(a, 0)| + |(0, b)| \geq |(a, 0) + (0, b)| = |(a, b)|$ for all vectors a and b. Therefore, we have that

$$\alpha_\ell\left(\frac{1}{N}\sum_{k=0}^{N-1} |(x(k), u(k))|\right) \geq \alpha_\ell\left(\frac{|(\mathbf{x}, \mathbf{u})|}{N}\right)$$

Finally, because $|b| \leq |(a,b)|$ for a vector (a,b), we have that

$$\alpha_\ell(|(\mathbf{x},\mathbf{u})|/N) \geq \alpha_\ell(|(x,\mathbf{u})|/N) := \alpha_1(|(x,\mathbf{u})|) \tag{35}$$

From (12) and (13) we have that $V_N(0,\mathbf{u}) \leq V_f(0) = 0$. We conclude that $\mathbf{u} = 0$ for all suboptimal input sequences under consideration and thus $\kappa_N(0,\tilde{\mathbf{u}}) = \{0\}$. □

Proposition 10. *There exists a function $\alpha_r(\cdot) \in \mathcal{K}_\infty$ such that $|\mathbf{u}| \leq \alpha_r(|x|)$ for any $(x,\mathbf{u}) \in \mathcal{Z}_r$.*

Proof. First consider the case $x \in r\mathbb{B}$. From (35), (13), (10), and because $|\mathbf{u}| \leq |(x,\mathbf{u})|$ we have that

$$\alpha_1(|\mathbf{u}|) \leq \alpha_1(|x,\mathbf{u}|) \leq V_N(x,\mathbf{u}) \leq V_f(x) \leq \alpha_f(|x|)$$

Therefore, $|\mathbf{u}| \leq \tilde{\alpha}_r(|x|)$ in which $\tilde{\alpha}_r(\cdot) := \alpha_1^{-1} \circ \alpha_f(\cdot)$ and $\tilde{\alpha}_r \in \mathcal{K}_\infty$.

Next we consider the case $x \notin r\mathbb{B}$. Define $\mu := \max_{\mathbf{u}\in\mathbb{U}^N} |\mathbf{u}|$ and note that μ is finite due to Assumption 2. Next, define $\gamma := \min\{1, \tilde{\alpha}_r(r)\} > 0$ and define $\alpha_r(\cdot) := (\mu/\gamma)\tilde{\alpha}_r(\cdot)$, which satisfies $|\mathbf{u}| \leq \alpha_r(|x|)$ for all $(x,\mathbf{u}) \in \mathcal{Z}_r$. □

Proposition 24. *If $\alpha_i(\cdot) \in \mathcal{K}_\infty$ for $i \in \mathbb{I}_{0:n}$, then $\min_i\{\alpha_i(\cdot)\} := \alpha_{\min}(\cdot) \in \mathcal{K}_\infty$.*

Proof. First, we demonstrate that for $\alpha_a(\cdot), \alpha_b(\cdot) \in \mathcal{K}_\infty$, the function $\min\{\alpha_a(\cdot), \alpha_b(\cdot)\} := \alpha_c(\cdot) \in \mathcal{K}_\infty$. Assume without loss of generality that for some x we have that $\alpha_c(x) = \alpha_a(x)$. Choose $y > x$. If $\alpha_c(y) = \alpha_a(y)$, we have that $\alpha_c(x) < \alpha_c(y)$. If $\alpha_c(y) = \alpha_b(y)$, we have that $\alpha_c(x) = \alpha_a(x) \leq \alpha_b(x) < \alpha_b(y) = \alpha_c(y)$. Therefore, because we have that $\alpha_c(x) < \alpha_c(y)$ for $x < y$, the function $\alpha_c(\cdot)$ is strictly increasing. Next, $\alpha_c(\cdot)$ is the minimum of two continuous functions and is therefore continuous. It follows from the definition of $\alpha_c(\cdot)$ that $\alpha_c(0) = 0$ and $\alpha_c(\cdot)$ is unbounded, so $\alpha_c(\cdot) \in \mathcal{K}_\infty$.

Next, we prove the general case by induction. Define $\tilde{\alpha}_i(\cdot) := \min\{\tilde{\alpha}_{i-1}(\cdot), \alpha_i(\cdot)\}$ with $\tilde{\alpha}_0(\cdot) := \alpha_0(\cdot)$. The function $\tilde{\alpha}_0(\cdot) \in \mathcal{K}_\infty$ by assumption, and by the above result, if the functions $\tilde{\alpha}_{i-1}(\cdot), \alpha_i(\cdot) \in \mathcal{K}_\infty$ then we have that $\tilde{\alpha}_i(\cdot) \in \mathcal{K}_\infty$. Finally, because $\tilde{\alpha}_n(\cdot) = \alpha_{\min}(\cdot)$, by induction we have that $\alpha_{\min}(\cdot) \in \mathcal{K}_\infty$. □

Proposition 13. *If the set $\mathcal{Z}$ is positive invariant for the difference inclusion $z^+ \in H(z)$, $H(0) = \{0\}$, $0 \in \mathcal{Z}$ and there exists a Lyapunov function $V(\cdot)$ on $\mathcal{Z}$, then the origin is asymptotically stable in $\mathcal{Z}$.*

Proof. This proof is similar to that of Theorem 12 in (Rawlings and Mayne, 2011). From (18) we have that $\alpha_2(|z|) \geq V(z)$ and therefore $|z| \geq \alpha_2^{-1}(V(z))$. Substituting this relation into (19), we have that

$$\begin{aligned}\sup_{z^+\in H(z)} V(z^+) &\leq V(z) - \alpha_3(|z|) \\ &\leq V(z) - \alpha_3(\alpha_2^{-1}(V(z))) \\ &= \sigma_1(V(z))\end{aligned}$$

in which $\sigma_1(\cdot) := (\cdot) - \alpha_3(\alpha_2^{-1}(\cdot))$. Using the properties of $\mathcal{K}$ functions (see e.g. (Khalil, 2002)), we conclude that $\sigma_1(\cdot)$ is continuous on its domain $\mathbb{R}_{\geq 0}$, zero at zero and that

$0 < \sigma_1(s) < s$ for $s > 0$. However, for $\sigma_1(\cdot)$ to be a $\mathcal{K}_\infty$ function, it must also be strictly increasing and unbounded. To satisfy these properties, we define

$$\sigma_2(s) = \max_{s' \in [0,s]} \sigma_1(s'), \quad s \in \mathbb{R}_{\geq 0}$$

The set $[0, s]$ is compact and $\sigma_1(\cdot)$ is continuous, so the maximum exists. By optimality, $\sigma_2(\cdot)$ is nondecreasing. Suppose $\sigma_2(\cdot)$ is discontinuous at $c \in \mathbb{R}_{\geq 0}$, i.e., has a positive jump. Define

$$a_1 := \lim_{s \nearrow c} \sigma_2(s), \quad a_2 := \lim_{s \searrow c} \sigma_2(s)$$

in which the limits exist because $\sigma_2(\cdot)$ is nondecreasing (Bartle and Sherbert, 2000, pp. 149-150). Because $\sigma_1(\cdot)$ is continuous, we must have that $\sigma_1(c) \leq a_1 < a_2$ or the limit of $\sigma_2(\cdot)$ from below is violated. Since $\sigma_1(c) < a_2$ and $\sigma_1(\cdot)$ is continuous, we must have $\sigma_1(s) = a_2$ for some $s < c$ or the limit from above is violated. However, $\sigma_1(s) = a_2$ for some $s < c$ violates the limit from below, and we therefore conclude that $\sigma_2(\cdot)$ is continuous by contradiction. We define

$$\sigma(s) = (1/2)(s + \sigma_2(s)), \quad s \in \mathbb{R}_{\geq 0}$$

in which now $\sigma(\cdot)$ is continuous, strictly increasing, and unbounded and $\sigma(0) = 0$. We conclude that $\sigma(\cdot) \in \mathcal{K}_\infty$ and note that $\sigma_1(s) < \sigma(s) < s$ for $s > 0$. Therefore

$$\sup_{z^+ \in H(z)} V(z^+) \leq \sigma(V(z))$$

Repeated use of the above and then (18) gives for all $i \geq 0$

$$V(z(i; z)) \leq \sigma^i(\alpha_2(|z|))$$

in which $\sigma^i(\cdot)$ represents the composition of $\sigma(\cdot)$ with itself i times and $z(i; z)$ is any solution to $z^+ \in H(z)$ at time i starting at $z \in \mathcal{Z}$. Using (18), we have that for all $i \geq 0$

$$|z(i; z)| \leq \beta(|z|, i)$$

in which

$$\beta(s, i) = \alpha_1^{-1}(\sigma^i(\alpha_2(s)))$$

To prove that $\beta(\cdot)$ is a $\mathcal{KL}$ function, we first note that for all $s \geq 0$, the sequence $w_i = \sigma^i(\alpha_2(s))$ is nonincreasing with i and bounded below by zero. The sequence w_i therefore converges, say, to a, as $i \to \infty$. By the definition of σ^i, we have that $w_i \to a$ and $\sigma(w_i) \to a$ as $a \to \infty$. Since $\sigma(\cdot)$ is continuous, we also have that $\sigma(w_i) \to \sigma(a)$ as $i \to \infty$. As a result, $\sigma(a) = a$, which implies that $a = 0$. Therefore, for all $s \geq 0$, $\beta(s, i) \to 0$ as $i \to \infty$. From the properties of $\mathcal{K}$ functions, we have that $\alpha_1^{-1}(\sigma^i(\alpha_2(s)))$ is a $\mathcal{K}$ function for all $i \geq 0$. We conclude that $\beta(\cdot) \in \mathcal{KL}$ and the proof is complete. □

Theorem 14. *Under Assumptions 1–4, a closed-loop system under Algorithm 9 is asymptotically stable in the set* $(x, \tilde{\mathbf{u}}) \in \mathcal{Z}_N$.

Proof. First we show that $V_N(z)$ is a Lyapunov function for (17) on $\mathcal{Z}_N$. From (35) we have that $\alpha_1(|z|) \leq V_N(z)$ for all $z \in \mathcal{Z}_N$. Therefore the lower bound condition in (18) of Definition 12 is satisfied. From (9), we conclude that the upper bound condition in (18) of Definition 12 is satisfied. By (11) and construction of the warm start (16) we have that $z^+ \in \mathcal{Z}_N$, so that $\mathcal{Z}_N$ is positive invariant.

As in standard MPC analysis, we have for all $z \in \mathcal{Z}_N$ that

$$V_N(z^+) \leq V_N(x, \mathbf{u}) - \ell(x, u(0))$$

due to (16), (12) and Assumption 3. From Assumption 4 we can write

$$V_N(z^+) \leq V_N(x, \mathbf{u}) - \alpha_\ell(|x, u(0)|)$$

Case I: $x \notin r\mathbb{B}$. We have that $\tilde{\mathbf{u}} \in \tilde{\mathcal{U}}_r(x)$ and from Proposition 10 we have that

$$\begin{aligned} |(x, \tilde{\mathbf{u}})| &\leq |x| + |\tilde{\mathbf{u}}| \leq |x| + \alpha_r(|x|) \\ &:= \alpha_{r*}(|x|) \leq \alpha_{r*}(|(x, u(0))|) \end{aligned} \tag{36}$$

Consequently,

$$\alpha_\ell(\alpha_{r*}^{-1}(|(x, \tilde{\mathbf{u}})|)) \leq \alpha_\ell(|(x, u(0))|)$$

Defining $\alpha_{3,1}(\cdot) := \alpha_\ell(\alpha_{r*}^{-1}(\cdot))$ and using (12), we have that

$$V_N(z^+) \leq V_N(x, \mathbf{u}) - \alpha_{3,1}(|z|) = V_N(z) - \alpha_{3,1}(|z|)$$

Case II: $x \in r\mathbb{B}$.
Case i: $(1/2)\alpha_1(|(x, \tilde{\mathbf{u}})|) \leq V_f(x)$. We have that

$$\begin{aligned} |(x, \tilde{\mathbf{u}})| &\leq \alpha_1^{-1}(2V_f(x)) \leq \alpha_1^{-1}(2\alpha_f(|x|)) \\ &\leq \alpha_1^{-1}(2\alpha_f(|(x, u(0))|)) \end{aligned}$$

We conclude that

$$V_N(z^+) \leq V_N(z) - \alpha_{3,2}(|z|)$$

in which $\alpha_{3,2}(\cdot) := \alpha_\ell(\alpha_f^{-1}(1/2\alpha_1(\cdot))$.

Case ii: $(1/2)\alpha_1(|(x, \tilde{\mathbf{u}})|) > V_f(x)$. We have that

$$\begin{aligned} V_N(x, \mathbf{u}) \leq V_f(x) &< (1/2)\alpha_1(|(x, \tilde{\mathbf{u}})|) \\ &\leq V_N(x, \tilde{\mathbf{u}}) - (1/2)\alpha_1(|(x, \tilde{\mathbf{u}})|) \end{aligned}$$

Therefore

$$V_N(z^+) \leq V_N(x, \mathbf{u}) \leq V_N(z) - \alpha_{3,3}(|z|)$$

with $\alpha_{3,3}(\cdot) := (1/2)\alpha_1(\cdot)$.

Let $\alpha_3(\cdot) = \min\{\alpha_{3,1}(\cdot), \alpha_{3,2}(\cdot), \alpha_{3,3}(\cdot)\}$. By Proposition 24, $\alpha_3(\cdot) \in \mathcal{K}_\infty$, and therefore

$$V_N(z^+) \leq V_N(z) - \alpha_3(|z|)$$

for all $z \in \mathcal{Z}_N$ and $z^+ \in H(z)$. We conclude that $V_N(z)$ is a Lyapunov function for (17) in $\mathcal{Z}_N$. Asymptotic stability follows directly from Proposition 13. □

7.3 Robust asymptotic stability of suboptimal MPC

Proposition 18. *If a difference inclusion $z^+ \in H_w(z)$ admits an ISS Lyapunov function on a positive invariant set $\mathcal{Z}$ for all $\|\mathbf{w}\| \leq \delta$ for some $\delta > 0$, then it is robustly asymptotically stable, as defined in Definition 16 for all $\|\mathbf{w}\| \leq \delta$.*

Proof. This proof is similar to that in Jiang and Wang (2001). It is not identical because, firstly, we consider a difference inclusion, rather than a difference equation, but because both the proof in this paper and in Jiang and Wang (2001) rely on cost-decrease, that difference is minor. Secondly, we consider a positive invariant set here rather than all of $\mathbb{R}^n$. Finally, we provide explicit construction of the function $\beta(\cdot)$.

We require some $\rho(\cdot) \in \mathcal{K}$ such that $\mathcal{C}_\rho := \text{lev}_{\rho(\|\mathbf{w}\|)} V$ is positive invariant. To find this $\rho(\cdot)$, we treat ρ as a positive constant and find how large $\|\mathbf{w}\|$ can be made while $\mathcal{C}_\rho$ remains positive invariant. Then we construct the function $\rho(\cdot)$. Pick $z \in \mathcal{C}_\rho$ We proceed by dividing $\mathcal{C}_\rho$ into outer and inner sublevel sets. In the outer set, we require that cost does not increase, but in the inner set we allow that cost may increase so long as it does not increase enough to leave the outer set.
Case I: $\rho/2 \leq V(z) \leq \rho$. We have that $V(z) \leq \alpha_2(|z|)$ and thus have $\alpha_2^{-1}(V(z)) \leq |z|$. Therefore, we have

$$\begin{aligned} V(z^+) \leq \sup_{z^+ \in H_w(z)} V(z^+) &\leq V(z) - \alpha_3(|z|) + \sigma(\|\mathbf{w}\|) \\ &\leq V(z) - \alpha_3(\alpha_2^{-1}(V(z))) + \sigma(\|\mathbf{w}\|) \\ &\leq V(z) - \alpha_3(\alpha_2^{-1}(\rho/2)) + \sigma(\|\mathbf{w}\|) \end{aligned}$$

Let $\alpha_4(\cdot) := \alpha_3(\alpha_2^{-1}(\cdot))$ and note that $\alpha_4(\cdot) \in \mathcal{K}_\infty$. Furthermore note that because $V(z^+) \leq V(z) - \alpha_4(V(z))$ for $\|\mathbf{w}\| = 0$ and because $V(\cdot)$ is nonnegative, $\alpha_4(s) \leq s$. In order to ensure cost does not increase, we require that $\sigma(\|\mathbf{w}\|) \leq \alpha_4(\rho/2)$. Thus we choose

$$\rho \geq 2\alpha_4^{-1}(\sigma(\|\mathbf{w}\|))$$

and therefore have

$$\begin{aligned} V(z^+) &\leq V(z) - \alpha_4(\rho/2) + \sigma(\|\mathbf{w}\|) \\ &\leq V(z) - \alpha_4(\alpha_4^{-1}(\sigma(\|\mathbf{w}\|))) + \sigma(\|\mathbf{w}\|) \\ &\leq V(z) - \sigma(\|\mathbf{w}\|) + \sigma(\|\mathbf{w}\|) = V(z) \end{aligned}$$

and cost cannot increase if $\rho/2 \leq V(z) \leq \rho$ and consequently $V(z^+) \leq \rho$.
Case II: $V(z) \leq \rho/2$. We have that

$$V(z^+) \leq V(z) - \alpha_3(|z|) + \sigma(\|\mathbf{w}\|) \leq \rho/2 + \sigma(\|\mathbf{w}\|)$$

Therefore, if $\rho/2 \geq \sigma(\|\mathbf{w}\|)$, then we conclude

$$V(z^+) \leq \rho/2 + \sigma(\|\mathbf{w}\|) \leq \rho/2 + \rho/2 = \rho$$

and that $V(z^+) \leq \rho$
So we have that $\mathcal{C}_\rho$ is positive invariant. Let $\rho(\cdot) := 2\max\{\alpha_4^{-1}(\sigma(\cdot)), \sigma(\cdot)\}$. By an argument similar to that of Proposition 24, $\rho(\cdot) \in \mathcal{K}$.

Next, we construct a $\mathcal{KL}$ function upper bound valid for $V(z) \geq \rho(\|\mathbf{w}\|)$.

$$\begin{aligned}V(z^+) \leq& V(z) - \alpha_4(V(z)) + \sigma(\|\mathbf{w}\|)\\ =& V(z) - \alpha_4(V(z)) + \alpha_4((1/2)V(z))\\ &- \alpha_4((1/2)V(z)) + \sigma(\|\mathbf{w}\|)\end{aligned}$$

As a result of $V(z) \geq \rho(\|\mathbf{w}\|)$, we have that

$$\begin{aligned}V(z^+) \leq& V(z) - \alpha_4(V(z)) + \alpha_4((1/2)V(z))\\ &- \alpha_4((1/2)\rho(\|\mathbf{w}\|)) + \sigma(\|\mathbf{w}\|)\end{aligned}$$

By the definition of $\rho(\cdot)$, we have that

$$\begin{aligned}V(z^+) \leq& V(z) - \alpha_4(V(z)) + \alpha_4((1/2)V(z))\\ &- \alpha_4((1/2)(2)\alpha_4^{-1}(\sigma(\|\mathbf{w}\|))) + \sigma(\|\mathbf{w}\|)\\ =& V(z) - \alpha_4(V(z)) + \alpha_4((1/2)V(z))\\ &- \sigma(\|\mathbf{w}\|) + \sigma(\|\mathbf{w}\|)\\ =& V(z) - (\alpha_4(V(z)) - \alpha_4((1/2)V(z)))\\ =& \tilde{\sigma}_V(V(z))\end{aligned}$$

in which $\tilde{\sigma}_V(s) := s - (\alpha_4(s) - \alpha_4(s/2))$. We have that $\tilde{\sigma}_V(\cdot)$ is continuous and zero at zero on $\mathbb{R}_{\geq}0$. It is nonnegative because, as we established earlier, $\alpha_4(s) \leq s$ and therefore $\alpha_4(s) - \alpha_4(s/2) \leq s$. Furthermore, because $\alpha_4(\cdot) \in \mathcal{K}_\infty$, we have that $\alpha_4(s) > \alpha_4(s/2)$ for $s > 0$ and therefore that $0 < \tilde{\sigma}_V(s) < s$ for $s > 0$. By the same process used in Proposition 13, we construct $\sigma_V \in \mathcal{K}_\infty$ such that $\tilde{\sigma}_V(s) \leq \sigma_V(s) < s$. Therefore, we can construct $\tilde{\beta}(s,k) := \sigma_V^k(s)$, in which $\sigma_V^k(\cdot)$ represents the composition of $\sigma_V(\cdot)$ with itself k times. As in Proposition 13, we have that $\tilde{\beta}(\cdot) \in \mathcal{KL}$.

We have $\tilde{\beta}(\cdot) \in \mathcal{KL}$ such that if $V(z(k)) \geq \rho(\|\mathbf{w}\|))$, then $V(z(k)) \leq \beta(V(z_0), k)$, and we also have that if $V(z(k)) \leq \rho(\|\mathbf{w}\|)$, then $V(z(k+1)) \leq \rho(\|\mathbf{w}\|)$. Therefore for all $z(0) \in \mathcal{Z}$, we have that

$$\alpha_1(|z(k)|) \leq V(z(k)) \leq \max\{\tilde{\beta}(V(z(0)), k), \rho(\|\mathbf{w}\|)\}$$

Because $\alpha_1^{-1}(\cdot)$ is nonnegative and strictly increasing, we have that

$$\begin{aligned}|z(k)| &\leq \max\{\alpha_1^{-1}(\tilde{\beta}(V(z(0)), k)), \alpha_1^{-1}(\rho(\|\mathbf{w}\|))\}\\ &\leq \alpha_1^{-1}(\tilde{\beta}(\alpha_2^{-1}(|z(0)|), k)) + \alpha_1^{-1}(\rho(\|\mathbf{w}\|))\end{aligned}$$

By defining $\alpha_1^{-1}(\tilde{\beta}(\alpha_2^{-1}(\cdot), \cdot)) := \beta(\cdot)$ and $\alpha_1^{-1}(\rho(\cdot)) := \gamma(\cdot)$, we have that $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|z(k)| \leq \beta(|z(0)|, k) + \gamma(\|\mathbf{w}\|)$$

Therefore the system is ISS. □

Proposition 19. *Let $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$, with $\mathcal{C}$ compact, $\mathcal{D}$ closed, and $f : \mathcal{D} \to \mathbb{R}^n$ continuous. Then there exists $\sigma(\cdot) \in \mathcal{K}_\infty$ such that for all $x \in \mathcal{C}$ and $y \in \mathcal{D}$, we have that $|f(x) - f(y)| \leq \sigma(|x-y|)$.*

Proof. The first part of this proof is similar to that of Theorem 4.19 (the Heine-Cantor theorem) in (Rudin, 1976, pp. 91). Choose $\epsilon > 0$. Because f is continuous, for all $x \in \mathcal{C}$ there exists $\delta_x > 0$ such that $|x-y| \leq \delta_x$ implies $|f(x) - f(y)| \leq \epsilon/2$. Let $\tilde{N}_x := \{y : |x-y| < \delta_x/2\}$. Note that $\{\tilde{N}_x\}$ is an open cover of $\mathcal{C}$. Because $\mathcal{C}$ is compact, there exists $\{x_i\}$ such that $\{\tilde{N}_{x_i}\}$ is a finite subcover of $\mathcal{C}$. Let $\{N_{x_i}\} := \{\{y : |x-y| \leq \delta_x/2\} \mid x \in \{x_i\}\}$ be the corresponding closed cover, and let $\delta = \min_i \delta_{x_i}/2$. Let $x \in \mathcal{C}$ and $y \in \mathcal{D}$. Then, for some i, $x \in N_{x_i}$. If $|x-y| \leq \delta$, then

$$|y - x_i| \leq |x - x_i| + |y - x| \leq \delta + \delta_{x_i}/2 \leq \delta_{x_i}$$

Therefore, by continuity, $|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(y) - f(x_i)| \leq \epsilon/2 + \epsilon/2 = \epsilon$.

For $x \in \mathcal{C}$ and $y \in \mathcal{D}$ we have that $|x-y| \leq \delta(\epsilon)$ implies $|f(x) - f(y)| \leq \epsilon$. By a natural extension of Proposition 15 in Rawlings and Risbeck (2015) there exists some $\sigma(\cdot) \in \mathcal{K}$ such that $|f(x) - f(y)| \leq \sigma(|x-y|)$. The function $\sigma(\cdot)$ can then be taken to be a member of $\mathcal{K}_\infty$. □

7.4 Example: discontinuous optimal cost

We consider the system described in Section 5:

$$x^+ = x\left(5u + \frac{1}{1 + (x-1)^2}\right)$$

in which x and u are scalars. We present an MPC control law for this system with a horizon length of two. We have both $u \in \mathbb{U} = [0,1]$ and $\mathcal{X}_f = [-0.5, 0.5]$. Furthermore, we have the following stage and terminal costs:

$$\ell(x,u) = x^2 + 10u$$
$$V_f(x) = 10x^2$$

We use dynamic programming to find the optimal control law. First, we solve for u_1^0 as a function of x_1.

$$\min_{u_1} V_1(x_1, u_1) = x_1^2 + 10u_1 + 10x_2^2 \tag{37}$$

We substitute (32) into the above to obtain:

$$V_1(x_1, u_1) = x_1^2 + 10u_1 + 10x_1^2\left(5u_1 + \frac{1}{1 + (x_1 - 1)^2}\right)^2$$

By differentiating with respect to u_1 we obtain:

$$\frac{dV_1}{du_1} = 10\left(1 + 10x_1^2\left(5u_1 + \frac{1}{1+(x_1-1)^2}\right)\right)$$

Because $u_1 \in [0,1]$, this expression is strictly positive for all x_1. Therefore, the smallest feasible u_1 is optimal.

As noted in Section 5, because $|f(x,u)| \geq |f(x,0)|$, it is only necessary to check $f(x,0) \in \mathcal{X}_f$ in order to establish whether the optimal control problem is feasible. Furthermore, because the sign of x^+ is the same as the sign of x, it is only necessary to check the upper bound for positive x_1 and the lower bound for negative x_1. So first, we require $-0.5 \leq x_2$ for negative x_1. By substituting (32) into this bound and setting $u_1 = 0$, we obtain

$$-0.5 \leq \frac{x_1}{1 + (x_1 - 1)^2}$$

By multiplying both sides by $1 + (x_1 - 1)^2$ and rearranging, we obtain

$$-0.5 - 0.5x_1^2 \leq 0$$

Therefore the lower bound of the terminal constraint is satisfied for all x_1.

Next, we examine the constraint $x_2 \leq 0.5$. By substituting (32) into it and setting $u_1 = 0$, we obtain

$$\frac{x_1}{1 + (x_1 - 1)^2} \leq 0.5$$

By multiplying both sides by $1 + (x_1 - 1)^2$ and rearranging, we obtain

$$0 \leq 1 - 2x_1 + 0.5x_1^2$$

This quadratic has roots $a_1 = 2 - \sqrt{2}$ and $b_1 = 2 + \sqrt{2}$. Therefore, the upper bound of the terminal constraint is satisfied for all $x_1 \in \mathcal{X}_1 = (-\infty, a_1] \cup [b_1, \infty)$.

Finally, by substituting the optimal control law into (37), we obtain the optimal cost

$$V_1^0(x_1) = \frac{10x_1^2}{1 + (x_1 - 1)^2}$$

For the next step of dynamic programming, we solve the following:

$$\min_{u_0} V_2(x_0, u_0) = x_0^2 + 10u_0 + V_1^0(x_1)$$

By substituting in for both V_1^0 and x_1, we obtain

$$V_2(x_0, u_0) = x_0^2 + 10u_0 + \frac{10x_0^2\left(5u_0 + \frac{1}{1+(x_0-1)^2}\right)^2}{1 + \left(x_0\left[5u_0 + \frac{1}{1+(x_0-1)^2}\right] - 1\right)^2}$$

By differentiating this expression with respect to u_0 we obtain

$$\begin{aligned}\frac{dV_2}{du_0} = \frac{5}{((x_0-1)^2+1)^3}\Big[&20x_0^2\left(5u_0\left[(x_0-1)^2+1\right]+1\right) \\ &+ 2x_0^2\left(5u_0\left[(x_0-1)^2+1\right]+1\right)\left((x_0-1)^2+1\right)^2 \\ &+ \left((x_0-1)^2+1\right)^3\Big]\end{aligned}$$

Note that all of the expressions in this derivative are nonnegative, with some that are strictly positive. Thus $dV_2/du_0 > 0$ for all feasible u_0, and therefore the smallest feasible u_0 is the optimal solution.

For recursive feasibility, we require

$$x_0 \left(5u_0 + \frac{1}{1+(x_0-1)^2}\right) \in (-\infty, a_1] \cup [b_1, \infty)$$

As before, for $x_0 \leq 0$ this requirement is satisfied by $u_0 = 0$, which is therefore the optimal input. First, we find which x_0 and u_0 result in $x_1 \in (-\infty, a_1]$. Because $|f(x,0)| \leq |f(x,u)|$, we only need to check $u_0 = 0$. Therefore, we require

$$\frac{x_0}{1+(x_0-1)^2} \leq a_1$$

By multiplying both sides by $1 + (x_0 - 1)^2$ and rearranging, we obtain

$$0 \leq a_1 x_0^2 - (2a_1 + 1) + 2a_1$$

This quadratic has two real roots a_2 and b_2. Note that $a_1 < a_2 < b_2 < b_1$. So for all $x \in (-\infty, a_2] \cap [b_2, \infty)$, we have that $u_0 = 0$ is both feasible and optimal.

Next, we check which x_0 and u_0 satisfy $x_1 \in [b_1, \infty)$. We require that

$$b_1 \leq x_0 \left(5u_0 + \frac{1}{1+(x_0-1)^2}\right)$$

By rearranging this expression, we obtain

$$\frac{1}{5}\left(\frac{b_1}{x_0} - \frac{1}{1+(x_0-1)^2}\right) \leq u_0$$

Now we check for which x_0 such an input is feasible. Because $u_0 \in [0,1]$ and $|f(x,u)|$ strictly increases with increased u, we set $u_0 = 1$ to obtain

$$b_1 \leq x_0 \left(5 + \frac{1}{1+(x_0-1)^2}\right)$$

By multiplying both sides by $1 + (x_0 - 1)^2$ and rearranging, we obtain

$$0 \leq x_0^3 + (b_1 - 2)x_0^2 + (2b_1 + 3)x_0 - 2b_1$$

This cubic has one real root, c. Because $c < a_2$, for all x_0 there exists a feasible u_0 and therefore $\mathcal{X}_2 = \mathbb{R}$. Furthermore, by choosing the smallest u_0 for each x_0, we obtain the following optimal control law:

$$u_2^0(x) = \begin{cases} \frac{1}{5}\left(\frac{b_1}{x} - \frac{1}{1+(x-1)^2}\right) & x \in (a_2, b_2) \\ 0 & \text{else} \end{cases}$$

which is what we set out to demonstrate.

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