On the equivalence between statements with $\epsilon$-$\delta$ and $K$-functions

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1 Introduction and Motivating Examples

The purpose of this note is to establish some simple results enabling direct translation between classic $\epsilon$-$\delta$ statements and $K$-function statements. The definition of $K$-function is standard.

**Definition 1 (K-function).** A $K$-function is a function defined on a nonempty interval $[0,b]$ with $b > 0$, $\gamma : [0,b] \rightarrow \mathbb{R}_{\geq 0}$ that is continuous, strictly increasing, and zero at zero.

Note that we require $\gamma(\cdot)$ to be defined only on some nonzero interval, not $[0,\infty)$.

As a motivating example, consider the standard $\epsilon$-$\delta$ definition of continuity of a function $f(\cdot)$ at a point $x$.

**Definition 2 (Continuity: $\epsilon$-$\delta$).** A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous at $x$ if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ (note that $\delta(\epsilon)$ may depend on $x$) such that

$$|f(x + p) - f(x)| \leq \epsilon \quad \text{for all } |p| \leq \delta(\epsilon) \quad (1)$$

The equivalent definition of continuity in the language of $K$-functions is the following.

**Definition 3 (Continuity: $K$-function).** A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous at $x$ if there exists a $K$-function $\gamma(\cdot)$ (note that the function $\gamma(\cdot)$ may depend on $x$) such that

$$|f(x + p) - f(x)| \leq \gamma(|p|) \quad \text{for all } |p| \in \text{Dom}(\gamma) \quad (2)$$

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To establish the equivalence of these definitions, we require the following result establishing a connection between the (possibly discontinuous) function $\delta(\epsilon)$ and existence of a $K$-function underbound.

**Proposition 4** (A $K$-function underbound of $\delta(\epsilon)$). Let $\delta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be an increasing, i.e., nondecreasing, function. Then there exists a $K$-function $\alpha(\cdot)$ such that for all $\epsilon > 0$

$$\alpha(\epsilon) \leq \delta(\epsilon)$$

**Proof.** In this proof, we construct the $K$-function $\alpha(\cdot)$ from the given function $\delta(\cdot)$. Figures 1–3 shows the techniques we employ. Start by taking an arbitrary $a_0 > 0$, and create a doubly infinite sequence, $a_i$ with $i = 0, \pm 1, \pm 2, \ldots$, such that $a_i$ is strictly increasing and tends to infinity and $a_{-i}$ is strictly decreasing and tends to zero as $i$ tends to infinity. We have that the $a_i$ sequence is strictly increasing. Now define the sequence $y_i$ by

$$y_i = \delta(a_{i-1}) \quad i = 0, \pm 1, \pm 2, \ldots$$

Note this right shift trick, depicted in Figure 1, is useful when creating an underbounding function. Since $\delta(\cdot)$ is a positive, increasing function, we have that $y_i = \delta(a_{i-1}) > 0$ and $y_i$ is an increasing sequence. Next define the continuous function $f(\cdot)$ by connecting the

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1 Note that we can assume $\delta(\epsilon)$ is an increasing function. See Proposition 11
Figure 2: Making an \( f'(\epsilon) \) function (solid line) that is strictly increasing with \( f'(0) = 0 \) from an increasing function \( f(\epsilon) \) (dashed line).

So far we have a function \( f(\cdot) \) defined on \([0, \infty)\) that is continuous (and piecewise linear), increasing, and satisfies \( f(\cdot) < \delta(\cdot) \). But \( f(\cdot) \) may not be a \( K \)-function because \( f(0) \) may not be zero, and \( f(\cdot) \) may not be strictly increasing. We next create a function with these properties. See also Figure 2.

If the function \( f(\cdot) \) is a constant function with value \( y_0 > 0 \), define \( \alpha(\epsilon) \) as any strictly increasing function starting at zero that underbounds \( y_0 \). For example

\[
\alpha(\epsilon) = y_0(1 - e^{-\epsilon})
\]

If \( f(\cdot) \) is not constant, take any index \( i_0 \) such that \( y_{i_0} < y_{i_0+1} \). For simplicity, relabel the \( a_i, y_i \) sequences such that \( i_0 = 0 \). Starting at \( i = 1 \), find the first set of indices (if any) \( i \in [i_1, i_2] \) where \( y_i \) is constant and \( y_{i_2} < y_{i_2+1} \). On such intervals define \( f'(\epsilon) \) to be the linear function

\[
f'(\epsilon) = \left( \frac{a_{i_2} - \epsilon}{a_{i_2} - a_{i_1}} \right) y_{i_1} + \left( \frac{\epsilon - a_{i_1}}{a_{i_2} - a_{i_1}} \right) y_{i_2}, \quad \epsilon \in [a_{i_1}, a_{i_2}]
\]

Note that \( f'(\cdot) \) is continuous, strictly increasing, and underbounds \( f(\cdot) \) on the interval \([a_{i_1}, a_{i_2}]\). Continue to the next interval of indices over which \( y_i \) is constant and repeat.

\[\text{We define } f(0) \text{ as } \lim_{\epsilon \searrow 0} f(\epsilon), \text{ which exists because } f(\cdot) \text{ is monotone.}\]
Figure 3: Treating the (usual) case when \( f(\epsilon) \) converges to zero as \( \epsilon \) converges to zero.

While increasing \( i \), if \( y_i \) becomes constant on an interval \([i_3, \infty)\) with \( y_{i_3-1} < y_{i_3} \), then create the underbound

\[
f'(\epsilon) = (y_{i_3} - y_{i_3-1})(1 - e^{-\epsilon_{i_3-1}}), \quad \epsilon \geq a_{i_3-1}
\]

In this fashion we have constructed an \( f'(\cdot) \) that is strictly increasing on \([a_0, \infty)\) and is an underbound of \( \delta(\cdot) \) on this interval.

Next we turn attention to the interval \([0, a_0]\). If \( f(\epsilon) \) converges to some \( b > 0 \) as \( \epsilon \to 0 \), then define \( f'(\epsilon) \) on \([0, a_1]\) as the linear function connecting the point \((0, 0)\) to \((a_0, b)\) and then join the function \( f'(\epsilon) \) for \( \epsilon \geq a_1 \) as shown in Figure 2.\(^3\) Setting \( \alpha(\cdot) = f'(\cdot) \), we then have a \( K \)-function underbound on \([0, \infty)\) for this case.

Finally, if \( f(\epsilon) \) converges to zero as \( \epsilon \to 0 \) (the usual case), proceed as in the previous part and replace intervals of constant values by their linear underbounds as shown in Figure 3.\(^4\) In this case also, setting \( \alpha(\cdot) = f'(\cdot) \), we have constructed a \( K \)-function underbound on \([0, \infty)\) and the proof is complete.

Note that the \( K \)-function \( \alpha(\cdot) \) is defined on \([0, \infty)\) and the \( K \)-function \( \alpha^{-1}(\cdot) \) is defined on \([0, \delta]\) in which \( \delta > 0 \) is any value satisfying \( \delta < \sup_{\epsilon>0} \delta(\epsilon) \).

**Proposition 5** (Equivalence of two continuity definitions). *The classic \( \epsilon-\delta \) definition and \( K \)-function definition of continuity are equivalent.*

**Proof.**

\(^3\)Note that we have now redefined \( f'(\epsilon) \) on the interval \([a_0, a_1]\).

\(^4\)Note that in this last case, unlike when treating the increasing \( a_i \) values, there is no interval \([0, a_{i_4}]\) on which \( f(\epsilon) \) can be constant because \( f(0) = 0 \) but \( f(a_{i_4}) > 0 \) for all \( i_4 \).
**K definition implies ε-δ definition.** Given the K-function $\gamma(\cdot)$ satisfying (2) choose $\delta(\epsilon) := \gamma^{-1}(\epsilon)$. We then have $|p| \leq \delta(\epsilon) = \gamma^{-1}(\epsilon)$ implies that $|f(x + p) - f(x)| \leq \gamma(|p|) \leq \gamma(\gamma^{-1}(\epsilon)) = \epsilon$ and the ε-δ definition of continuity is established.

**ε-δ definition implies K definition.** Since $\alpha(\cdot)$ defined in Proposition 4 is defined on $[0, \infty)$, for any $\epsilon > 0$ choose $p$ so that $|p| = \alpha(\epsilon)$. Since $|p| = \alpha(\epsilon) \leq \delta(\epsilon)$, by ε-δ continuity, we have that $|f(x + p) - f(x)| \leq \epsilon = \alpha^{-1}(|p|)$. Note that $\alpha^{-1}(\cdot)$ is a K-function defined on $[0, \delta]$ and the K-function definition of continuity is established.

As a second example, consider the definition of Lyapunov stability.

**Definition 6** (Lyapunov stability: ε-δ). Consider the dynamic system $x^+ = f(x)$ with $f(0) = 0$. The origin is Lyapunov stable if for every $\epsilon > 0$ the exists $\delta(\epsilon) > 0$ such that for $|x| \leq \delta$, the solution satisfies for all $k \geq 0$

$$|\phi(k; x)| \leq \epsilon$$

The equivalent definition with a K-function is the following.

**Definition 7** (Lyapunov stability: K-function). Consider the dynamic system $x^+ = f(x)$ satisfying $f(0) = 0$. The origin is Lyapunov stable if there exists a K-function $\gamma(\cdot)$ such that for all $k \geq 0$

$$|\phi(k; x)| \leq \gamma(|x|)$$

As a third example, consider the definition of robust global asymptotic stability in ε-δ language.

**Definition 8** (Robust global asymptotic stability: ε-δ). Consider a globally asymptotically stable nominal system $x^+ = f(x)$. The perturbed system $x^+ = f(x) + w$ is robustly globally asymptotically stable if there exists a KL-function $\beta(\cdot)$ and for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for all $\|w\| \leq \delta, x \in \mathbb{R}^n$, and $k \geq 0$

$$|x(k; x, w)| \leq \beta(|x|, k) + \epsilon$$

The equivalent K-function definition is the following.

**Definition 9** (Robust global asymptotic stability: K-function). Consider a globally asymptotically stable nominal system $x^+ = f(x)$. The perturbed system $x^+ = f(x) + w$ is robustly globally asymptotically stable if there exists a K-function $\gamma(\cdot)$ and KL-function $\beta(\cdot)$ such that for all $x \in \mathbb{R}^n$ and $k \geq 0$

$$|x(k; x, w)| \leq \beta(|x|, k) + \gamma(\|w\|)$$

Note that we could write the final inequality equivalently as

$$|x(k; x, w)| \leq \beta(|x|, k) + \gamma(\|w\|_{0:k-1})$$

because $x(k; x, w)$ depends on $w$ only up to time $k - 1$. The last statement is equivalent to the statement that $x^+ = f(x) + w$ is input-to-state stable (ISS) considering the disturbance $w$ as the input.
2 Generalization

The following definitions and theorem generalize the previous examples. Let $X$ be any normed space.

**Definition 10** (Property $P$). A system with testable condition $C : X \to \mathbb{R}_{\geq 0}$ satisfying $C(0) = 0$ has property $P$ if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$C(x) \leq \epsilon \quad \text{for every } x \in X \text{ satisfying } |x| \leq \delta(\epsilon)$$

We note that the function $\delta(\epsilon)$ in Definition 10 can be made increasing, as shown in the following proposition.

**Proposition 11** ($\delta(\epsilon)$ can be made increasing). Suppose a system has property $P$ as in Definition 10. Then, without loss of generality, the function $\delta(\epsilon)$ can be assumed to be a nondecreasing function.

**Proof.** Suppose (10) holds for $\hat{\delta}(\epsilon)$ which is possibly not nondecreasing. Next, define $\delta(\epsilon) := \min(\hat{\delta}(\epsilon), 1)$. We note that (10) holds also for $\delta(\epsilon)$ because $(0, \delta(\epsilon)) \subseteq (0, \hat{\delta}(\epsilon))$, and thus (10) holding for $\delta$ is weaker than for $\hat{\delta}$. Then, define

$$\delta(\epsilon) := \frac{1}{2} \sup_{s \in (0, \epsilon]} \delta(s)$$

which is well-defined because $\delta(s) \in (0, 1)$ for all $s > 0$, and all bounded sets of real numbers have suprema. Furthermore, $\delta(\epsilon)$ is clearly nondecreasing. To show that (10) holds for $\delta(\epsilon)$, let $\epsilon_1 > 0$ be arbitrary. By definition, there exists positive $\epsilon_0 < \epsilon_1$ such that $\bar{\delta}(\epsilon_0) \geq \delta(\epsilon_1)$.

Thus, from (10) and these two inequalities, we know that for arbitrary $x \in X$,

$$|x| \leq \delta(\epsilon_1) \implies |x| \leq \delta(\epsilon_0) \implies C(x) \leq \epsilon_0 \implies C(x) \leq \epsilon_1$$

which means (10) holds for $\delta(\epsilon)$ and the statement is proved. ■

In the language of $K$-functions, we have the following definition of property $P_K$.

**Definition 12** (Property $P_K$). A system with testable condition $C : X \to \mathbb{R}_{\geq 0}$ satisfying $C(0) = 0$ has property $P_K$ if there exists $b > 0$ and $K$-function $\gamma(\cdot)$ defined on $[0, b]$, such that for all $x \in X$ satisfying $|x| \leq b$

$$C(x) \leq \gamma(|x|)$$

**Proposition 13** (Equivalence of $P$ and $P_K$). A system has property $P$ if and only if it has property $P_K$. The constant $b$ defined in property $P_K$ can be chosen as any positive value satisfying $b < \sup_{\epsilon > 0} \delta(\epsilon)$ with $\delta(\epsilon)$ defined in property $P$.

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5Suppose not. Then, for all $s \in (0, \epsilon]$, we have $\bar{\delta}(s) < \delta(s) < \sup_{s \in (0, \epsilon]} \delta(s)$, which is a contradiction because we have found an upper bound strictly less than the supremum.
3 Extensions

Here we show how a global $K$-function can be found for a locally bounded function.

**Proposition 14** (Global $K$-function overbound.) Let $X \subseteq \mathbb{R}^n$ be closed and suppose that a function $V : X \to \mathbb{R}_{\geq 0}$ is continuous at $x_0 \in X$ and locally bounded on $X$ (i.e., bounded on every compact subset of $X$). Then, there exists a $K$-function $\alpha$ such that

$$|V(x) - V(x_0)| \leq \alpha(|x - x_0|) \text{ for all } x \in X$$

**Proof.** First, by Proposition 5, we know that there exists a local overbounding function, i.e., there exists a $K$-function $\gamma$ and a constant $a > 0$ such that

$$|V(x) - V(x_0)| \leq \gamma(|x - x_0|) \text{ whenever } |x - x_0| \leq b_0$$

Note that any $b_0 \in \text{Dom}(\gamma)$ will suffice.

From here, we proceed similarly to Proposition 11 in Rawlings and Mayne (2011). Starting from $b_0$, choose any strictly increasing sequence $\{b_i\}$. For each $i \in \mathbb{I}_{\geq 1}$, let $B_i = \{x \in X : |x - x_0| \leq b_i\}$. We note that each $B_i$ is a compact subset of $X$. Next, define a sequence $\{\beta_i\}$ as

$$\beta_i := \sup_{x \in B_i} |V(x) - V(x_0)| + i$$

which is well-defined by compactness of the $B_i$. We note also that the $\beta_i$ are strictly increasing. Finally, define

$$\alpha(s) := \begin{cases} \frac{\beta_1}{\gamma(b_0)} \gamma(s) & s \in [0, b_0) \\ \beta_i + (\beta_{i+2} - \beta_i) \frac{s - b_i}{b_{i+1} - b_i} & s \in [b_i, b_{i+1}) \text{ for all } i \in \mathbb{I}_{\geq 0} \end{cases}$$

We illustrate this construction in Figure 4. Clearly, $\alpha(0) = 0$ and $\alpha$ is continuous and increasing. Furthermore, because we have shifted the $\beta_i$ as before, we see that $|V(x) - V(x_0)| \leq \alpha(|x - x_0|)$.

We note that for the case of $V(x) \geq 0$ and $x_0 = 0$, we have

$$V(x) \leq \alpha(|x|) \text{ for all } x \in X$$

and thus, $\alpha$ gives a global overbound.

**References**

Figure 4: Construction of $\alpha$. The function $\alpha(s)$ is constructed by rescaling $\gamma(s)$ on $[0, b_0]$ (green) and then linearly interpolating (red) the points $(b_i, \beta_{i+1})$ (blue).