Addendum to the paper “Is suboptimal nonlinear MPC inherently robust?”

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Abstract

In this report we present detailed proofs of the results presented in the paper “Is suboptimal nonlinear MPC inherently robust?” [1], which were omitted in the published paper due to space limitations.

Keywords

Nonlinear MPC, Suboptimal solutions, Robust stability, Lyapunov functions

1 Statements and omitted proofs

We report in the statement of some results and the proofs omitted from the paper “Is suboptimal nonlinear MPC inherently robust?” [1]. All numberings are referred to the paper [1].

Proposition 5. Any $u_0^\gamma(x^+)$, optimal solution to $P_N(x^+)$, satisfies conditions (3a)–(3b) for all $x^+ \in X_N$. Moreover, if $x^+ \in X_f$ condition (3c) is satisfied by $u^0(x^+)$.  

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Thus, choosing $V$ for inclusion Proposition 7. We have that $\kappa_N(0) = \{0\}$ and $F(0) = \{0\}$.

Proof: From $\ell(x, u) \geq a_1|(x, u)|^a$ and from $V_f(x) \geq 0$, we have

$$V_N(x, u) \geq a_1' \sum_{k=0}^{N-1} |(\phi(k; x, u), u(k))|^a \geq a_1' \left[ |(x, u(0))|^a + \sum_{k=1}^{N-1} |u(k)|^a \right] \geq a_1' N^{-a} |(x, u)|^a,$$

Thus, choosing $a_1$ to satisfy $0 < a_1 \leq N^{-a} a_1'$, we have that $V_N(x, u) \geq a_1|(x, u)|^a$ for all $(x, u) \in \mathbb{X} \times \mathbb{U}^N$. From (3c), Assumptions 1 and 4, we have that $a_1' |(0, u)|^a \leq V_N(0, u) \leq V_f(0) = 0$. Thus, it follows that $u = 0$ and hence $\kappa_N(0) = u(0; 0) = \{0\}$. The second result then follows from Assumption 1.

**Proposition 10.** If $V$ is an exponential Lyapunov function on the set $\mathcal{Z}$ for the difference inclusion $z^+ \in H(z)$, there exists $0 < \gamma < 1$ such that:

$$\max_{z^+ \in H(z)} V(z^+) \leq \gamma V(z)$$

Proof: From the definition of $V$, $z \in \mathcal{Z}$ implies that

$$\max_{z^+ \in H(z)} V(z^+) \leq V(z) - a_3 |z|^a \leq V(z) - \frac{a_3}{a_2} V(z) \leq \gamma V(z)$$

for $\gamma > 1 - a_3/a_2$. Since $a_2 \geq a_3 > 0$, we have $0 < \gamma < 1$.

**Lemma 11.** If the set $\mathcal{Z}$, $0 \in \mathcal{Z}$, is positively invariant for the difference inclusion $z^+ \in H(z)$, $H(0) = \{0\}$, and there exists an exponential Lyapunov function $V$ on $\mathcal{Z}$ the origin is ES on $\mathcal{Z}$.

Proof: Since $\psi(k; z) \in \mathcal{Z}$ for all $k \in \mathbb{I}_{\geq 0}$, and using Proposition 10, we write:

$$|\psi(k; z)|^a \leq \frac{V(\psi(k; z))}{a_1} \leq \frac{\lambda^k V(z)}{a_1} \leq \frac{\lambda}{a_1} a_2 |z|^a. \quad \text{Thus, we obtain: } |\psi(k; z)| \leq b\lambda^k |z| \text{ in which } \lambda = \gamma^{1/a} \text{ and } b = \left( \frac{\gamma^a}{a_1} \right)^{1/a}, \text{ and we note that } 0 < \lambda < 1.$$  

**Lemma 12.** There exists a positive constant $c$ such that $|u| \leq c|x|$ for any $(x, u) \in \mathcal{Z}$. 

Proof:
Proof: We first show that $|u| \leq \bar{c}|x|$ holds, for some $\bar{c}$, if $x \in rB \subseteq X_f$. Recall from the proof of Proposition 7 that there is $a_1 > 0$ such that $a_1 |(x,u)|^\alpha \leq V_N(x,u)$ for all $(x,u) \in X \times U_N$. For $x \in rB \subseteq X_f \subseteq X$, we can therefore write:

$$a_1 |u|^\alpha \leq a_1 |(x,u)|^\alpha \leq V_N(x,u) \leq V_f(x) \leq a_f|x|^\alpha$$

Thus, given any $\bar{c} \geq (a_f/a_1)^{(1/\alpha)}$ we obtain $|u| \leq \bar{c}|x|$ for any $x \in rB$. Define $\mu = \max_{u \in U_N} |u|$, and note that $\mu < \infty$ because $U_N$ is compact. Choosing $c \geq \max\{\bar{c}, c\}$, we observe that $|u| \leq c|x|$ for all $(x,u) \in Z_r$. In fact, if $x \in rB$ we have that $|u| \leq \bar{c}|x| \leq c|x|$; while if $x \notin rB$ we have that $|u| \leq \mu \leq \frac{\mu|x|}{r} \leq c|x|$.

Lemma 13. $V_N(z)$ is an exponential Lyapunov function for the extended closed-loop system (5) in any compact subset of $Z_r$.

Proof: As established in the proof of Proposition 7, we have that $a_1 |z|^\alpha \leq V_N(z)$ for some $a_1 > 0$ and all $z \in Z_r$. Consider any compact set $C \subseteq Z_r$ and define: $\mu = \max_{z \in C} V_N(z)$. Note that from Assumption 1, it follows that $V_N(\cdot)$ is continuous; thus, $\mu$ is well defined. From Assumption 4, if we choose $a_2 \geq \max\{\mu/r^a, a'_2\}$, we have that:

$$V_N(z) \leq a_2 |z|^a \quad \text{for all } z \in C.$$ 

We verify this fact by noting that if $z \in rB \cap Z_r$, we have from Assumption 4 that $V_N(z) \leq a'_2 |z|^a \leq a_2 |z|^a$; if instead $z \in Z_r \setminus rB$ we have that $V_N(z) \leq \mu \leq |z|^a/r^a \leq a_2 |z|^a$. We now prove that $V_N(z^+) \leq V_N(z) - a_3 |z|^a$ for all $z^+ \in H(z)$ and $z \in Z_r$. In fact, for all $z^+ \in H(z)$ we have from Assumption 4 that

$$V_N(z^+) \leq V_N(z) - \ell(x,u(0)) \leq V_N(z) - a'_1 |(x,u(0))|^a.$$

From Lemma 12 we can write:

$$|z| \leq (|x| + |u|) \leq (1 + c)|x| \leq (1 + c)|(x,u(0))|$$

Thus, if we define a positive constant $a_3 \leq \frac{a'_1}{(1+c)^a}$, we can write:

$$V_N(z^+) \leq V_N(z) - a'_1 |(x,u(0))|^a \leq V_N(z) - \frac{a'_1}{(1+c)^a} |z|^a \leq V_N(z) - a_3 |z|^a$$

for all $z^+ \in H(z)$ and $z \in Z_r$. □

Lemma 20. For every $\mu > 0$, there exists a $\delta > 0$ such that, for all $(z_m,e,d,e^+)$ in $Z_r \times \delta B \times \delta B \times \delta B$, $z = z_m - (e,0)$, such that $x^+_m \in X_N$, and some $\gamma$, $0 < \gamma < 1$, we have:

$$\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \max\{\gamma V_N(z), \mu\}$$

Proof: Let $\mu > 0$ be given. The value $V_N(\bar{x}^+, \bar{u})$ is the cost along the nominal trajectory (no disturbance). Therefore since $V_N(\cdot)$ is an exponential Lyapunov function for the nominal system (Lemma 13), Proposition 10 gives that

$$V_N(\bar{x}^+, \bar{u}) \leq V_N(z_m) - \ell(x_m, u(0); x_m) \leq \gamma V_N(z_m)$$
for some $0 < \gamma < 1$. Consider a $\gamma$ such that $\gamma < \gamma < 1$, and define $\rho = \mu(\gamma - \gamma) > 0$. Recall that: $\hat{x}^{+} - x_{m}^{+} = f(x_{m}, u(0; x_{m})) - f(x, u(0; x_{m})) - d - e^{+}$. Due to continuity of $V_{N}$ and $f$, and because of $|p| \leq \sigma(|\hat{x}^{+} - x_{m}^{+}|)$, we can choose $\delta_{1} > 0$ such that the following condition holds for all $(z_{m}, e, d, e^{+}) \in \mathcal{Z}_{r} \times \delta_{1} \mathbb{B} \times \delta_{1} \mathbb{B} \times \delta_{1} \mathbb{B}$, $z = z_{m} - (e, 0)$:

$$V_{N}(x_{m}^{+}, \bar{u} + p) \leq V_{N}(\hat{x}^{+}, \bar{u}) + \frac{\rho}{3}.$$ 

(1)

By continuity of $V_{N}$, choose $\delta_{2} > 0$ such that the condition:

$$V_{N}(\hat{x}^{+}, \bar{u}) \leq \gamma V_{N}(x_{m}, u) \leq \gamma V_{N}(x, u) + \frac{\rho}{3}$$

(2)

holds for all $(z_{m}, e) \in \mathcal{Z}_{r} \times \delta_{2} \mathbb{B}$, $z = z_{m} - (e, 0)$. From continuity of $V_{N}$ and $f$ and from (11b), choose $\delta_{3} > 0$ such that

$$V_{N}(x^{+}, u^{+}) \leq V_{N}(x_{m}^{+}, u^{+}) + \frac{\rho}{3} \leq V_{N}(x_{m}^{+}, \bar{u} + p) + \frac{\rho}{3}$$

(3)

for all $z^{+} = (x^{+}, u^{+}) \in H_{ed}(z)$ and all $(z_{m}, e, d, e^{+}) \in \mathcal{Z}_{r} \times \delta_{3} \mathbb{B} \times \delta_{3} \mathbb{B} \times \delta_{3} \mathbb{B}$, $z = z_{m} - (e, 0)$. Defining $\delta = \min\{\delta_{1}, \delta_{1}, \delta_{3}\}$, and summing up (the most external sides of) (1)–(3), we obtain:

$$\max_{z^{+} \in H_{ed}(z)} V_{N}(z^{+}) \leq \gamma V_{N}(z) + \rho$$

for all $(z_{m}, e, d, e^{+}) \in \mathcal{Z}_{r} \times \delta \mathbb{B} \times \delta \mathbb{B} \times \delta \mathbb{B}$, $z = z_{m} - (e, 0)$. Define $\mathcal{Z}_{1} = \{z = z_{m} - (e, 0) \mid z_{m} \in Z_{r}, e \in \delta \mathbb{B}, V_{N}(z) \leq \mu\}$ and $\mathcal{Z}_{2} = \{z = z_{m} - (e, 0) \mid z_{m} \in Z_{r}, e \in \delta \mathbb{B}, V_{N}(z) > \mu\}$, and assume that $\mu$ is not so large that $\mathcal{Z}_{2}$ is empty (otherwise the proof is simpler). If $z \in \mathcal{Z}_{2}$ we can write: $\max_{z^{+} \in H_{ed}(z)} V_{N}(z^{+}) \leq \gamma V_{N}(z) + \rho \leq \gamma V_{N}(z) + \mu(\gamma - \gamma) \leq \mu$. If instead $z \in \mathcal{Z}_{2}$ we can write: $\max_{z^{+} \in H_{ed}(z)} V_{N}(z^{+}) \leq \gamma V_{N}(z) + \mu(\gamma - \gamma) \leq \gamma V_{N}(z)$. Therefore, we have established that the condition:

$$\max_{z^{+} \in H_{ed}(z)} V_{N}(z^{+}) \leq \max\{\gamma V_{N}(z), \mu\}$$

holds for all $(z_{m}, e, d, e^{+}) \in \mathcal{Z}_{r} \times \delta \mathbb{B} \times \delta \mathbb{B} \times \delta \mathbb{B}$, $z = z_{m} - (e, 0)$, such that $x_{m}^{+} \in X_{N}$. \hspace{1cm} \square

References