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Addendum to the paper "Is suboptimal nonlinear MPC inherently robust?"

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Abstract

In this report we present detailed proofs of the results presented in the paper "Is suboptimal nonlinear MPC inherently robust?" [1], which were omitted in the published paper due to space limitations.

Keywords

Nonlinear MPC, Suboptimal solutions, Robust stability, Lyapunov functions

1 Statements and omitted proofs

We report in the statement of some results and the proofs omitted from the paper "Is suboptimal nonlinear MPC inherently robust?" [1]. All numberings are referred to the paper [1].

Proposition 5. *Any* $\mathbf{u}^0(x^+)$, optimal *solution to* $\mathbb{P}_N(x^+)$, *satisfies conditions* (3a)–(3b) *for all* $x^+ \in \mathcal{X}_N$. *Moreover, if* $x^+ \in \mathcal{X}_f$ *condition* (3c) *is satisfied by* $\mathbf{u}^0(x^+)$.

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Proof: Satisfaction of (3a) and (3b) by $\mathbf{u}^0(x^+)$ is implied by the optimality of $\mathbf{u}^0(x^+)$. For the final claim, consider any $x^+ \in \mathbb{X}_f$, define $x(0) = x^+$ and choose any $u(0) \in \kappa_f(x(0))$ satisfying Assumption 3. We thus obtain $V_f(x(1)) + \ell(x(0), u(0)) \leq V_f(x(0))$. Because $x(1) \in \mathbb{X}_f$, we can choose $u(1) \in \kappa_f(x(1))$ satisfying Assumption 3 to obtain $V_f(x(2)) + \ell(x(1), u(1)) + \ell(x(0), u(0)) \leq V_f(x(1)) + \ell(x(0), u(0)) \leq V_f(x(0))$. Continuing in this fashion for $k = 2, 3, \ldots, N-1$, and defining $\mathbf{u}_f = (u(0), u(1), \ldots, u(N-1))$, we obtain $V_N(x^+, \mathbf{u}_f) \leq V_f(x^+)$. Finally, optimality of $\mathbf{u}^0(x^+)$ implies that (3c) holds for $\mathbf{u}^0(x^+)$. □

Proposition 7. *We have that* $\kappa_N(0) = \{0\}$ *and* $F(0) = \{0\}$.

Proof: From $\ell(x, u) \geq a_1' |(x, u)|^a$ and from $V_f(x) \geq 0$, we have

$$V_N(x, \mathbf{u}) \geq a_1' \sum_{k=0}^{N-1} |(\phi(k; x, \mathbf{u}), u(k))|^a$$

$$\geq a_1' \left[|(x, u(0))|^a + \sum_{k=1}^{N-1} |u(k)|^a \right] \geq a_1' N^{-a} |(x, \mathbf{u})|^a,$$

Thus, choosing a_1 to satisfy $0 < a_1 \leq N^{-a} a_1'$, we have that $V_N(x, \mathbf{u}) \geq a_1 |(x, \mathbf{u})|^a$ for all $(x, \mathbf{u}) \in \mathbb{X} \times \mathbb{U}^N$. From (3c), Assumptions 1 and 4, we have that $a_1 |(0, \mathbf{u})|^a \leq V_N(0, \mathbf{u}) \leq V_f(0) = 0$. Thus, it follows that $\mathbf{u} = 0$ and hence $\kappa_N(0) = u(0; 0) = \{0\}$. The second result then follows from Assumption 1. □

Proposition 10. *If* V *is an* exponential Lyapunov *function on the set* $\mathcal{Z}$ *for the difference inclusion* $z^+ \in H(z)$, *there exists* $0 < \gamma < 1$ *such that:*

$$\max_{z^+ \in H(z)} V(z^+) \leq \gamma V(z)$$

Proof: From the definition of V, $z \in \mathcal{Z}$ implies that

$$\max_{z^+ \in H(z)} V(z^+) \leq V(z) - a_3 |z|^a \leq V(z) - \tfrac{a_3}{a_2} V(z) \leq \gamma V(z)$$

for $\gamma > 1 - a_3/a_2$. Since $a_2 \geq a_3 > 0$, we have $0 < \gamma < 1$. □

Lemma 11. *If the set* $\mathcal{Z}$, $0 \in \mathcal{Z}$, *is positively invariant for the difference inclusion* $z^+ \in H(z)$, $H(0) = \{0\}$, *and there exists an exponential Lyapunov function* V *on* $\mathcal{Z}$ *the origin is ES on* $\mathcal{Z}$.

Proof: Since $\psi(k; z) \in \mathcal{Z}$ for all $k \in \mathbb{I}_{\geq 0}$, and using Proposition 10, we write: $|\psi(k; z)|^a \leq \frac{V(\psi(k;z))}{a_1} \leq \frac{\gamma^k V(z)}{a_1} \leq \frac{\gamma^k a_2 |z|^a}{a_1}$. Thus, we obtain: $|\psi(k; z)| \leq b \lambda^k |z|$ in which $\lambda = \gamma^{1/a}$ and $b = \left(\frac{a_2}{a_1}\right)^{1/a}$, and we note that $0 < \lambda < 1$. □

Lemma 12. *There exists a positive constant* c *such that* $|\mathbf{u}| \leq c|x|$ *for any* $(x, \mathbf{u}) \in \mathcal{Z}_r$.

Proof: We first show that $|\mathbf{u}| \leq \bar{c}|x|$ holds, for some $\bar{c}$, if $x \in r\mathbb{B} \subseteq \mathbb{X}_f$. Recall from the proof of Proposition 7 that there is $a_1 > 0$ such that $a_1|(x, \mathbf{u})|^a \leq V_N(x, \mathbf{u})$ for all $(x, \mathbf{u}) \in \mathbb{X} \times \mathbb{U}^N$. For $x \in r\mathbb{B} \subseteq \mathbb{X}_f \subseteq \mathbb{X}$, we can therefore write:

$$a_1|\mathbf{u}|^a \leq a_1|(x, \mathbf{u})|^a \leq V_N(x, \mathbf{u}) \leq V_f(x) \leq a_f|x|^a$$

Thus, given any $\bar{c} \geq (a_f/a_1)^{(1/a)}$ we obtain $|\mathbf{u}| \leq \bar{c}|x|$ for any $x \in r\mathbb{B}$. Define $\mu = \max_{\mathbf{u} \in \mathbb{U}^N} |\mathbf{u}|$, and note that $\mu < \infty$ because $\mathbb{U}^N$ is compact. Choosing $c \geq \max\{\frac{\mu}{r}, \bar{c}\}$, we observe that $|\mathbf{u}| \leq c|x|$ for all $(x, \mathbf{u}) \in \mathcal{Z}_r$. In fact, if $x \in r\mathbb{B}$ we have that $|\mathbf{u}| \leq \bar{c}|x| \leq c|x|$; while if $x \notin r\mathbb{B}$ we have that $|\mathbf{u}| \leq \mu \leq \frac{\mu|x|}{r} \leq c|x|$. □

Lemma 13. *$V_N(z)$ is an exponential Lyapunov function for the extended closed-loop system (5) in any compact subset of $\mathcal{Z}_r$.*

Proof: As established in the proof of Proposition 7, we have that $a_1|z|^a \leq V_N(z)$ for some $a_1 > 0$ and all $z \in \mathcal{Z}_r$. Consider any compact set $\mathcal{C} \subseteq \mathcal{Z}_r$ and define: $\mu = \max_{z \in \mathcal{C}} V_N(z)$. Note that from Assumption 1, it follows that $V_N(\cdot)$ is continuous; thus, μ is well defined. From Assumption 4, if we choose $a_2 \geq \max\{\mu/\bar{r}^a, a_2'\}$, we have that:

$$V_N(z) \leq a_2|z|^a \quad \text{for all} \quad z \in \mathcal{C}.$$

We verify this fact by noting that if $z \in \bar{r}\mathbb{B} \cap \mathcal{Z}_r$, we have from Assumption 4 that $V_N(z) \leq a_2'|z|^a \leq a_2|z|^a$; if instead $z \in \mathcal{Z}_r \setminus \bar{r}\mathbb{B}$ we have that $V_N(z) \leq \mu \leq \mu|z|^a/\bar{r}^a \leq a_2|z|^a$. We now prove that $V_N(z^+) \leq V_N(z) - a_3|z|^a$ for all $z^+ \in H(z)$ and $z \in \mathcal{Z}_r$. In fact, for all $z^+ \in H(z)$ we have from Assumption 4 that

$$V_N(z^+) \leq V_N(z) - \ell(x, u(0)) \leq V_N(z) - a_1'|(x, u(0))|^a.$$

From Lemma 12 we can write:

$$|z| \leq (|x| + |\mathbf{u}|) \leq (1 + c)|x| \leq (1 + c)|(x, u(0))|$$

Thus, if we define a positive constant $a_3 \leq \frac{a_1'}{(1+c)^a}$, we can write:

$$V_N(z^+) \leq V_N(z) - a_1'|(x, u(0))|^a \leq V_N(z) - \frac{a_1'}{(1+c)^a}|z|^a \leq V_N(z) - a_3|z|^a$$

for all $z^+ \in H(z)$ and $z \in \mathcal{Z}_r$. □

Lemma 20. *For every $\mu > 0$, there exists a $\delta > 0$ such that, for all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}$, $z = z_m - (e, 0)$, such that $x_m^+ \in \mathcal{X}_N$, and some γ, $0 < \gamma < 1$, we have:*

$$\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \max\{\gamma V_N(z), \mu\}$$

Proof: Let $\mu > 0$ be given. The value $V_N(\tilde{x}^+, \tilde{\mathbf{u}})$ is the cost along the nominal trajectory (no disturbance). Therefore since $V_N(\cdot)$ is an exponential Lyapunov function for the nominal system (Lemma 13), Proposition 10 gives that

$$V_N(\tilde{x}^+, \tilde{\mathbf{u}}) \leq V_N(z_m) - \ell(x_m, u(0; x_m)) \leq \bar{\gamma} V_N(z_m)$$

for some $0 < \bar{\gamma} < 1$. Consider a γ such that $\bar{\gamma} < \gamma < 1$, and define $\rho = \mu(\gamma - \bar{\gamma}) > 0$. Recall that: $\tilde{x}^+ - x_m^+ = f(x_m, u(0; x_m)) - f(x, u(0; x_m)) - d - e^+$. Due to continuity of V_N and f, and because of $|\mathbf{p}| \leq \sigma(|\tilde{x}^+ - x_m^+|)$, we can choose $\delta_1 > 0$ such that the following condition holds for all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta_1\mathbb{B} \times \delta_1\mathbb{B} \times \delta_1\mathbb{B}$, $z = z_m - (e, 0)$:

$$V_N(x_m^+, \tilde{\mathbf{u}} + \mathbf{p}) \leq V_N(\tilde{x}^+, \tilde{\mathbf{u}}) + \frac{\rho}{3}. \tag{1}$$

By continuity of V_N, choose $\delta_2 > 0$ such that the condition:

$$V_N(\tilde{x}^+, \tilde{\mathbf{u}}) \leq \bar{\gamma} V_N(x_m, \mathbf{u}) \leq \bar{\gamma} V_N(x, \mathbf{u}) + \frac{\rho}{3} \tag{2}$$

holds for all $(z_m, e) \in \mathcal{Z}_r \times \delta_2\mathbb{B}$, $z = z_m - (e, 0)$. From continuity of V_N and f and from (11b), choose $\delta_3 > 0$ such that

$$V_N(x^+, \mathbf{u}^+) \leq V_N(x_m^+, \mathbf{u}^+) + \frac{\rho}{3} \leq V_N(x_m^+, \tilde{\mathbf{u}} + \mathbf{p}) + \frac{\rho}{3} \tag{3}$$

for all $z^+ = (x^+, \mathbf{u}^+) \in H_{ed}(z)$ and all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta_3\mathbb{B} \times \delta_3\mathbb{B} \times \delta_3\mathbb{B}$, $z = z_m - (e, 0)$. Defining $\delta = \min\{\delta_1, \delta_1, \delta_3\}$, and summing up (the most external sides of) (1)–(3), we obtain:

$$\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \bar{\gamma} V_N(z) + \rho$$

for all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}$, $z = z_m - (e, 0)$. Define $\mathcal{Z}_1 = \{z = z_m - (e, 0) \mid z_m \in \mathcal{Z}_r,\ e \in \delta\mathbb{B},\ V_N(z) \leq \mu\}$ and $\mathcal{Z}_2 = \{z = z_m - (e, 0) \mid z_m \in \mathcal{Z}_r,\ e \in \delta\mathbb{B},\ V_N(z) > \mu\}$, and assume that μ is not so large that $\mathcal{Z}_2$ is empty (otherwise the proof is simpler). If $z \in \mathcal{Z}_1$ we can write: $\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \bar{\gamma} V_N(z) + \rho \leq \bar{\gamma}\mu + \mu(\gamma - \bar{\gamma}) \leq \mu$. If instead $z \in \mathcal{Z}_2$ we can write: $\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \bar{\gamma} V_N(z) + \mu(\gamma - \bar{\gamma}) \leq \gamma V_N(z)$. Therefore, we have established that the condition:

$$\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \max\{\gamma V_N(z), \mu\}$$

holds for all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}$, $z = z_m - (e, 0)$, such that $x_m^+ \in \mathcal{X}_N$. □

References

[1] G. Pannocchia, J. B. Rawlings, and S. J. Wright. Is suboptimal nonlinear MPC inherently robust? In *Proceedings of 18th IFAC World Congress*, 2011.