TWCCC ⋆ Texas – Wisconsin – California Control Consortium

Technical report number 2011-01

Addendum to the paper "Inherently robust suboptimal nonlinear MPC: theory and application"

Gabriele Pannocchia*
Dept. Chem. Eng., Ind. Chem. & Sc. Mat.
Univ. of Pisa. (ITALY)

James B. Rawlings†
Dept. Chem. & Biol. Eng.
Univ. of Wisconsin. (USA)

Stephen J. Wright‡
Comp. Sc. Dept.
Univ. of Wisconsin. (USA)

September 1, 2011

Abstract

In this report we present detailed proofs of the results presented in the paper "Inherently robust suboptimal nonlinear MPC: theory and application" [1], which were omitted in the published paper due to space limitations.

Keywords

Nonlinear MPC, Suboptimal solutions, Robust stability, Lyapunov functions

1 Statements and omitted proofs

We report in the statement of some results and the proofs omitted from the paper "Inherently robust suboptimal nonlinear MPC: theory and application" [1]. All numberings are referred to the paper [1].

*Email: `g.pannocchia@diccism.unipi.it`. Author to whom all correspondence should be addressed.
†Email: `rawlings@engr.wisc.edu`
‡Email: `swright@cs.wisc.edu`

Proposition 5. *Any* optimal *solution* $\mathbf{u}^0(x^+)$ *to* $\mathbb{P}_N(x^+)$, *satisfies conditions* (4a), (4b) *for all* $x^+ \in \mathcal{X}_N$. *Moreover, the inequality in condition* (4c) *is satisfied by* $\mathbf{u}^0(x^+)$ *for all* $x^+ \in \mathbb{X}_f$.

Proof: Satisfaction of (4a) and (4b) by $\mathbf{u}^0(x^+)$ is implied by the optimality of $\mathbf{u}^0(x^+)$. For the final claim, consider any $x^+ \in \mathbb{X}_f$, define $x(0) = x^+$ and choose any $u(0)$ satisfying Assumption 3 for $x(0)$. We thus obtain $V_f(x(1)) + \ell(x(0), u(0)) \leq V_f(x(0))$. Because $x(1) \in \mathbb{X}_f$, we can choose $u(1)$ satisfying Assumption 3 for $x(1)$ to obtain $V_f(x(2)) + \ell(x(1), u(1)) + \ell(x(0), u(0)) \leq V_f(x(1)) + \ell(x(0), u(0)) \leq V_f(x(0))$. Continuing in this fashion for $k = 2, 3, \ldots, N-1$, and defining $\mathbf{u}_f = (u(0), u(1), \ldots, u(N-1))$, we obtain $V_N(x^+, \mathbf{u}_f) \leq V_f(x^+)$. Finally, optimality of $\mathbf{u}^0(x^+)$ implies that (4c) holds for $\mathbf{u}^0(x^+)$. □

Proposition 7. *We have that* $\kappa_N(0) = \{0\}$ *and* $F(0) = \{0\}$.

Proof. From $\ell(x, u) \geq a_1' |(x, u)|^a$ and from $V_f(x) \geq 0$, we have

$$V_N(x, \mathbf{u}) \geq a_1' \sum_{k=0}^{N-1} |(\phi(k; x, \mathbf{u}), u(k))|^a \geq a_1' \left[|(x, u(0))|^a + \sum_{k=1}^{N-1} |u(k)|^a \right] \geq a_1' N^{-a} |(x, \mathbf{u})|^a,$$

Thus, choosing a_1 to satisfy $0 < a_1 \leq N^{-a} a_1'$, we have that $V_N(x, \mathbf{u}) \geq a_1 |(x, \mathbf{u})|^a$ for all $(x, \mathbf{u}) \in \mathbb{X} \times \mathbb{U}^N$. From (4c), Assumptions 1 and 4, we have that $a_1 |(0, \mathbf{u})|^a \leq V_N(0, \mathbf{u}) \leq V_f(0) = 0$. Thus, it follows that $\mathbf{u} = 0$ and hence $\kappa_N(0) = u(0; 0) = \{0\}$. The second result then follows from Assumption 1. □

Proposition 10. *If* V *is an* exponential Lyapunov *function on the set* $\mathcal{Z}$ *for the difference inclusion* $z^+ \in H(z)$, *there exists* $0 < \gamma < 1$ *such that:*

$$\max_{z^+ \in H(z)} V(z^+) \leq \gamma V(z).$$

Proof. From the definition of V, $z \in \mathcal{Z}$ implies that

$$\max_{z^+ \in H(z)} V(z^+) \leq V(z) - a_3 |z|^a \leq V(z) - \tfrac{a_3}{a_2} V(z) \leq \gamma V(z)$$

for $\gamma > 1 - a_3/a_2$. Since $a_2 \geq a_3 > 0$, we have $0 < \gamma < 1$. □

Lemma 11. *If the set* $\mathcal{Z}$, $0 \in \mathcal{Z}$, *is positively invariant for the difference inclusion* $z^+ \in H(z)$, $H(0) = \{0\}$, *and there exists an exponential Lyapunov function* V *on* $\mathcal{Z}$, *the origin is ES on* $\mathcal{Z}$.

Proof. Since $\psi(k; z) \in \mathcal{Z}$ for all $k \in \mathbb{I}_{\geq 0}$, and using Proposition 10, we write: $|\psi(k; z)|^a \leq \frac{V(\psi(k;z))}{a_1} \leq \frac{\gamma^k V(z)}{a_1} \leq \frac{\gamma^k a_2 |z|^a}{a_1}$. Thus, we obtain: $|\psi(k; z)| \leq b \lambda^k |z|$ in which $\lambda = \gamma^{1/a}$ and $b = \left(\frac{a_2}{a_1}\right)^{1/a}$, and we note that $0 < \lambda < 1$. □

Lemma 12. *There exists a positive constant* c *such that* $|\mathbf{u}| \leq c|x|$ *for any* $(x, \mathbf{u}) \in \mathcal{Z}_r$.

Proof. We first show that $|\mathbf{u}| \leq \bar{c}|x|$ holds, for some $\bar{c}$, if $x \in r\mathbb{B} \subseteq \mathbb{X}_f$. Recall from the proof of Proposition 7 that there is $a_1 > 0$ such that $a_1|(x, \mathbf{u})|^a \leq V_N(x, \mathbf{u})$ for all $(x, \mathbf{u}) \in \mathbb{X} \times \mathbb{U}^N$. For $x \in r\mathbb{B} \subseteq \mathbb{X}_f \subseteq \mathbb{X}$, we can write:

$$a_1|\mathbf{u}|^a \leq a_1|(x, \mathbf{u})|^a \leq V_N(x, \mathbf{u}) \leq V_f(x) \leq a_f|x|^a.$$

Thus, given any $\bar{c} \geq (a_f/a_1)^{(1/a)}$, we obtain $|\mathbf{u}| \leq \bar{c}|x|$ for any $x \in r\mathbb{B}$. Define $\mu = \max_{\mathbf{u} \in \mathbb{U}^N} |\mathbf{u}|$, and note that $\mu < \infty$ because $\mathbb{U}^N$ is compact. Choosing $c \geq \max\{\frac{\mu}{r}, \bar{c}\}$, we observe that $|\mathbf{u}| \leq c|x|$ for all $(x, \mathbf{u}) \in \mathcal{Z}_r$. In fact, if $x \in r\mathbb{B}$ we have that $|\mathbf{u}| \leq \bar{c}|x| \leq c|x|$; while if $x \notin r\mathbb{B}$ we have that $|\mathbf{u}| \leq \mu \leq \frac{\mu|x|}{r} \leq c|x|$. □

Lemma 13. *$V_N(z)$ is an exponential Lyapunov function for the extended closed-loop system* (6) *in any compact subset of $\mathcal{Z}_r$.*

Proof. As established in the proof of Proposition 7, we have that $a_1|z|^a \leq V_N(z)$ for some $a_1 > 0$ and all $z \in \mathcal{Z}_r$. Consider any compact set $\mathcal{C} \subseteq \mathcal{Z}_r$ and define: $\mu = \max_{z \in \mathcal{C}} V_N(z)$. Note that from Assumption 1, it follows that $V_N(\cdot)$ is continuous; thus, μ is well defined. From Assumption 4, if we choose $a_2 \geq \max\{\mu/\bar{r}^a, a_2'\}$, we have that:

$$V_N(z) \leq a_2|z|^a \quad \text{for all} \quad z \in \mathcal{C}.$$

We verify this fact by noting that if $z \in \bar{r}\mathbb{B} \cap \mathcal{Z}_r$, we have from Assumption 4 that $V_N(z) \leq a_2'|z|^a \leq a_2|z|^a$; if instead $z \in \mathcal{Z}_r \setminus \bar{r}\mathbb{B}$ we have that $V_N(z) \leq \mu \leq \mu|z|^a/\bar{r}^a \leq a_2|z|^a$. We now prove that $V_N(z^+) \leq V_N(z) - a_3|z|^a$ for all $z^+ \in H(z)$ and $z \in \mathcal{Z}_r$. In fact, for all $z^+ \in H(z)$ we have from Assumption 4 that

$$V_N(z^+) \leq V_N(z) - \ell(x, u(0)) \leq V_N(z) - a_1'|(x, u(0))|^a.$$

From Lemma 12 we can write:

$$|z| \leq (|x| + |\mathbf{u}|) \leq (1 + c)|x| \leq (1 + c)|(x, u(0))|$$

Thus, if we define a positive constant $a_3 \leq \frac{a_1'}{(1+c)^a}$, we can write:

$$V_N(z^+) \leq V_N(z) - a_1'|(x, u(0))|^a \leq V_N(z) - \tfrac{a_1'}{(1+c)^a}|z|^a \leq V_N(z) - a_3|z|^a$$

for all $z^+ \in H(z)$ and $z \in \mathcal{Z}_r$. □

Theorem 14. *Under Assumptions 1, 2, 3, and 4, the origin of the closed-loop system* (5) *is ES on (arbitrarily large) compact subsets of $\mathcal{X}_N$.*

Proof. From Lemma 13 we have that $V_N(z)$ is an exponential Lyapunov function for (6) in any given compact subset of $\mathcal{Z}_r$. Let $\bar{V}$ be an arbitrary positive scalar, and consider the set

$$\mathcal{S} = \{(x, \mathbf{u}) \in \mathcal{Z}_r \mid V_N(x, \mathbf{u}) \leq \bar{V}\}.$$

We observe that $\mathcal{S} \subseteq \mathcal{Z}_r$ is compact and is invariant for (6). By Lemma 11, these facts prove that the origin of the extended system (6) is ES on $\mathcal{S}$, i.e., there exist scalars $b' > 0$ and $0 < \lambda < 1$, such that for any $z \in \mathcal{S}$ we can write:

$$\psi(k; z) \in \mathcal{S} \quad \text{and} \quad |\psi(k; z)| \leq b'\lambda^k |z| \qquad \text{for all } k \in \mathbb{I}_{\geq 0}$$

in which $\psi(k; z) = z(k)$ is *a solution* of (6) at time k for a given initial extended state $z(0) = z$. We define $\mathcal{C} = \{x \in \mathcal{X}_N \mid \exists \mathbf{u} \in \mathcal{U}_N(x) \text{ such that } (x, \mathbf{u}) \in \mathcal{S}\}$ and note that $\mathcal{C} \subseteq \mathcal{X}_N$ and that $\mathcal{C}$ is compact because it is the projection onto $\mathbb{R}^n$ of the compact set $\mathcal{S}$. Thus for any $x \in \mathcal{C}$ and its associated suboptimal input sequence $\mathbf{u}$ such that $z = (x, \mathbf{u}) \in \mathcal{S}$, we denote with $\phi(k; x)$ the state component of $\psi(k; z)$ (i.e., a solution of the nonextended system (5)). For all $k \in \mathbb{I}_{\geq 0}$ we can write:

$$\phi(k; x) \in \mathcal{C} \quad \text{and} \quad |\phi(k; x)| \leq |\psi(k; z)| \leq b'\lambda^k |z| \leq b\lambda^k |x|$$

in which $b = b'(1 + c)$, because from Lemma 12 it follows that $|z| \leq |x| + |\mathbf{u}| \leq (1 + c)|x|$. This inequality establishes that the origin of the closed-loop system (5) is ES on $\mathcal{C}$, and $\bar{V}$ can be chosen large enough for $\mathcal{C}$ to contain any given compact subset of $\mathcal{X}_N$. □

Proposition 19. *Under Assumption 18, for any* $(\tilde{x}^+, \tilde{\mathbf{u}}) \in \mathbb{Z}_N$ *and* $x_m^+ \in \mathcal{X}_N$*, the set of solutions to* (11) *is nonempty.*

Proof. The result follows directly from Assumption 18 by noticing that $\tilde{\mathbf{u}} \in \mathcal{U}_N(\tilde{x}^+)$ and $\tilde{\mathbf{u}} + \mathbf{p} \in \mathcal{U}_N(x_m^+)$. □

Lemma 20. *For every* $\mu > 0$*, there exists a* $\delta > 0$ *and* $\gamma \in (0, 1)$ *such that, for all* $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}$ *with* $x_m^+ \in \mathcal{X}_N$*, we have:*

$$\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \max\{\gamma V_N(z), \mu\},$$

where $z = z_m - (e, 0)$*.*

Proof. Let $\mu > 0$ be given. The value $V_N(\tilde{x}^+, \tilde{\mathbf{u}})$ is the cost along the nominal trajectory (no disturbance). Therefore since (by Lemma 13) $V_N(\cdot)$ is an exponential Lyapunov function for the nominal system, Proposition 10 gives that

$$V_N(\tilde{x}^+, \tilde{\mathbf{u}}) \leq V_N(z_m) - \ell(x_m, u(0; x_m)) \leq \bar{\gamma} V_N(z_m)$$

for some $0 < \bar{\gamma} < 1$. Consider a γ such that $\bar{\gamma} < \gamma < 1$, and define $\rho = \mu(\gamma - \bar{\gamma}) > 0$. Recall that: $\tilde{x}^+ - x_m^+ = f(x_m, u(0; x_m)) - f(x, u(0; x_m)) - d - e^+$. Due to continuity of V_N and f, and because of $|\mathbf{p}| \leq \sigma(|\tilde{x}^+ - x_m^+|)$, we can choose $\delta_1 > 0$ such that the following condition holds for all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta_1\mathbb{B} \times \delta_1\mathbb{B} \times \delta_1\mathbb{B}$, $z = z_m - (e, 0)$:

$$V_N(x_m^+, \tilde{\mathbf{u}} + \mathbf{p}) \leq V_N(\tilde{x}^+, \tilde{\mathbf{u}}) + \tfrac{\rho}{3}. \tag{1}$$

By continuity of V_N, choose $\delta_2 > 0$ such that

$$V_N(\tilde{x}^+, \tilde{\mathbf{u}}) \leq \bar{\gamma} V_N(x_m, \mathbf{u}) \leq \bar{\gamma} V_N(x, \mathbf{u}) + \tfrac{\rho}{3} \tag{2}$$

holds for all $(z_m, e) \in \mathcal{Z}_r \times \delta_2\mathbb{B}$, $z = z_m - (e, 0)$. From continuity of V_N and f and from (12b), choose $\delta_3 > 0$ such that

$$V_N(x^+, \mathbf{u}^+) \leq V_N(x_m^+, \mathbf{u}^+) + \tfrac{\rho}{3} \leq V_N(x_m^+, \tilde{\mathbf{u}} + \mathbf{p}) + \tfrac{\rho}{3} \tag{3}$$

for all $z^+ = (x^+, \mathbf{u}^+) \in H_{ed}(z)$ and all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta_3\mathbb{B} \times \delta_3\mathbb{B} \times \delta_3\mathbb{B}$, $z = z_m - (e, 0)$. Defining $\delta = \min\{\delta_1, \delta_1, \delta_3\}$, and summing up (the most external sides of) (1), (2), and (3), we obtain:

$$\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \bar{\gamma} V_N(z) + \rho$$

for all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}$, $z = z_m - (e, 0)$. Define

$$\mathcal{Z}_1 = \{z = z_m - (e, 0) \mid z_m \in \mathcal{Z}_r,\ e \in \delta\mathbb{B},\ V_N(z) \leq \mu\},$$
$$\mathcal{Z}_2 = \{z = z_m - (e, 0) \mid z_m \in \mathcal{Z}_r,\ e \in \delta\mathbb{B},\ V_N(z) > \mu\},$$

and assume that μ is not so large that $\mathcal{Z}_2$ is empty (otherwise the proof is simpler). If $z \in \mathcal{Z}_1$ we can write: $\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \bar{\gamma} V_N(z) + \rho \leq \bar{\gamma}\mu + \mu(\gamma - \bar{\gamma}) \leq \mu$. If instead $z \in \mathcal{Z}_2$ we can write: $\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \bar{\gamma} V_N(z) + \mu(\gamma - \bar{\gamma}) \leq \gamma V_N(z)$. Therefore, we have established that the condition:

$$\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \max\{\gamma V_N(z), \mu\}$$

holds for all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}$, $z = z_m - (e, 0)$, such that $x_m^+ \in \mathcal{X}_N$. □

Theorem 21. *Under Assumptions 1, 2, 3, 4, and 18, the origin of the perturbed closed-loop system* (7) *is SRES on* $\mathcal{C}_\rho$*.*

Proof. (Robust feasibility) Suppose that $x \in \mathcal{C}_\rho$ and let $z = (x, \mathbf{u})$ be the corresponding augmented state where $\mathbf{u}$ is a suboptimal sequence computed for the measured state $x_m = x + e$, $e \in \rho\mathbb{B}$. We recall that $V_N(z) \leq \bar{V}$, i.e., $z \in \mathcal{S}$ and that $z_m \in \mathcal{S}_\rho \subseteq \mathcal{Z}_r$. Moreover, we define $\tilde{z}^+ = (\tilde{x}^+, \tilde{\mathbf{u}})$. Since $V_N(\cdot)$ is an exponential Lyapunov function for the nominal system and $z_m \in \mathcal{Z}_r$, Proposition 10 gives that $V_N(\tilde{z}^+) \leq \bar{\gamma} V_N(z_m)$ for some $0 < \bar{\gamma} < 1$. Because $z_m \in \mathcal{S}_\rho$ it follows that $V_N(\tilde{x}^+, \tilde{\mathbf{u}}) \leq \bar{\gamma}\bar{V} < \bar{V}$. Recalling that: $\tilde{x}^+ - x_m^+ = f(x_m, u(0; x_m)) - f(x, u(0; x_m)) - d - e^+$, it follows from continuity of f that there exists a $\bar{\delta}_1 > 0$ such that $V_N(x_m^+, \tilde{\mathbf{u}}) < \bar{V}$ and thus $x_m^+ \in \mathcal{X}_N$ for all $(z_m, e, d, e^+) \in \mathcal{S}_\rho \times \bar{\delta}_1 \times \bar{\delta}_1 \times \bar{\delta}_1$. Hence, the initialization step (11) is well defined. Define any $0 < \mu < (1 - \bar{\gamma})\bar{V}$. From continuity of V_N and f, and because $|\mathbf{p}| \leq \sigma(|\tilde{x}^+ - x_m^+|)$, we can choose $\bar{\delta}_2 > 0$ such that the following condition holds: $V_N(x_m^+, \tilde{\mathbf{u}} + \mathbf{p}) \leq V_N(\tilde{x}^+, \tilde{\mathbf{u}}) + \mu < V_N(\tilde{x}^+, \tilde{\mathbf{u}}) + (1 - \bar{\gamma})\bar{V} \leq \bar{V}$. This proves that $V_N(z_m^+) \leq V_N(x_m^+, \tilde{\mathbf{u}} + \mathbf{p}) < \bar{V}$. From continuity of V_N it also follows that we can choose $\rho > 0$ sufficiently small that $V_N^\rho(z_m^+) \leq \bar{V}$. Taking $\delta = \min\{\rho, \bar{\delta}_1, \bar{\delta}_2\}$ we have proved that $z_m^+ \in \mathcal{S}_\rho$ for all $(z_m, e, d, e^+) \in \mathcal{S}_\rho \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}$. This implies:

$$x(k) \in \mathcal{C}_\rho \subseteq \mathcal{X}_N \quad \text{for all } k \in \mathbb{I}_{\geq 0}$$

and also that $x_m(k) \in \mathcal{X}_N$ for all $k \in \mathbb{I}_{\geq 0}$. Hence, (10a) holds.

(Robust stability) We denote by $\psi_{ed}(k;z)$ *a solution* of the perturbed difference inclusion (13) at time $k \in \mathbb{I}_{\geq 0}$ starting from the initial state $z(0) = z$ and given disturbance and measurement error sequences $\{d(k)\}$, $\{e(k)\}$. As established in the proof of Proposition 7 we have that there exists a scalar $a_1 > 0$ such that $a_1|z|^a \leq V_N(z)$ for any $z \in \mathcal{C}_\rho \times \mathbb{U}^N \subseteq \mathbb{X} \times \mathbb{U}^N$. Moreover, by Lemma 13 there exists a scalar $a_2 > 0$ such that $V_N(z) \leq a_2|z|^a$ for any $z \in \mathcal{C}_\rho \times \mathbb{U}^N$. From Lemma 20, by induction, we now write:

$$a_1|\psi_{ed}(k;z)|^a \leq V_N(\psi_{ed}(k;z)) \leq \max\{\gamma^k V_N(z), \mu\} \leq \max\{\gamma^k a_2|z|^a, \mu\}$$

which implies

$$|\psi_{ed}(k;z)| \leq \max\{\bar{b}\lambda^k|z|, (\mu/a_1)^{1/a}\} \leq \max\{\bar{b}\lambda^k|z|, \bar{\epsilon}\}$$

in which $\lambda = \gamma^{1/a}$, $\bar{b} = (a_2/a_1)^{1/a}$ and $\bar{\epsilon} = (\mu/a_1)^{1/a}$. Finally, from Lemma 12 recalling that $|\mathbf{u}| \leq c|x_m| \leq c|x| + c\delta$ and that $\phi_{ed}(k;x)$ represents the state component of $\psi_{ed}(k;z)$, for all $x \in \mathcal{C}_\rho$ we write:

$$|\phi_{ed}(k;x)| \leq |\psi_{ed}(k;z)| \leq \max\{\bar{b}\lambda^k|z|, \bar{\epsilon}\} \leq \bar{b}\lambda^k|z| + \bar{\epsilon} \leq b\lambda^k|x| + \epsilon$$

with $b = \bar{b}(1+c)$ and $\epsilon = \bar{\epsilon} + \bar{b}c\delta$, completing the proof because $0 < \lambda < 1$. □

Corollary 22. *Under Assumptions 1, 2, 3, 4, and 18, the origin of the perturbed closed-loop system* (7) *is RES on* $\mathrm{int}(\mathcal{X}_N)$.

Proof. This result follows immediately because robust feasibility is assumed in RES. Thus, for any compact set $\mathcal{C} \subset \mathrm{int}\,\mathcal{X}_N$, the second part (Robust stability) of the proof of Theorem 21 can be readily applied to obtain for all $x \in \mathcal{C}$:

$$|\phi_{ed}(k;x)| \leq b\lambda^k|x| + \epsilon \quad \text{for all } k \in \mathbb{I}_{\geq 0}$$

□

Lemma 24. $V_N^\beta(z)$ *is an exponential Lyapunov function for the extended closed-loop system* (6) *in any compact subset of* $\bar{\mathcal{Z}}_r$.

Proof. Following the proof of Lemma 12 we have again that there exists a constant c such that for any $(x, \mathbf{u}) \in \bar{\mathcal{Z}}_r$ the condition $|\mathbf{u}| \leq c|x|$ holds. Now consider the difference inclusion $z^+ \in H(z)$, where $H(z)$ is defined in (6) with $G(z)$ appropriately modified according to (20). Following the proof of Lemma 13, it follows again that for any $z = (x, \mathbf{u}) \in \bar{\mathcal{Z}}_r$ the conditions:

$$a_1|z|^a \leq V_N^\beta(z) \leq a_2|z|^a, \qquad \max_{z^+ \in H(z)} V_N^\beta(z^+) \leq V_N^\beta(z) - a_3|z|^a$$

hold for some positive scalars a_1, a_2, a_3. Hence, V_N^β is an exponential Lyapunov function in the set $\bar{\mathcal{Z}}_r$ for (6). □

Theorem 25. *Under Assumptions 1, 23, 3, and 4, the origin of the closed-loop system* (5) *is ES on* $\mathbb{X}_0$.

Proof. We first show that $\bar{\mathcal{Z}}_r$ is positively invariant for $z^+ \in H(z)$. To this aim, assume that $z \in \bar{\mathcal{Z}}_r$ and consider any $z^+ \in H(z)$. From (20b) it follows that:

$$V_N^\beta(z^+) = \sum_{k=0}^{N-1} \ell(\phi(k; x^+, \mathbf{u}^+)) + \beta V_f(\phi(N; x^+, \mathbf{u}^+)) \leq V_N^\beta(x^+, \tilde{\mathbf{u}}) \leq V_N^\beta(z) \leq \bar{V}$$

Since $\beta = \max\{1, \bar{V}/\alpha\}$ and by non-negativeness of $\ell(\cdot)$, it follows that $V_f(\phi(N; x^+, \mathbf{u}^+)) \leq \alpha$, i.e., $\phi(N; x^+, \mathbf{u}^+) \in \mathbb{X}_f$. Combining this, (20a) and (20c), it follows that $z^+ \in \bar{\mathcal{Z}}_r$. Thus, from Lemmas 24 and 11 the origin of the extended closed-loop system (6) is ES on $\bar{\mathcal{Z}}_r$, i.e., all solutions $\psi(k; z)$ satisfy (for some $b' > 0$ and $0 < \lambda < 1$) the conditions: $\psi(k; z) \in \bar{\mathcal{Z}}_r$ and $|\psi(k; z)| \leq b'\lambda^k |z|$ at all times $k \geq \mathbb{I}_{\geq 0}$. Finally, for any $x \in \mathbb{X}_0$ and any input sequence $\mathbf{u}$ such that $(x, \mathbf{u}) \in \bar{\mathcal{Z}}_r$, let $\phi(k; x)$ denote the state component of $\psi(k; z)$. It follows that, at all times $k \in \mathbb{I}_{\geq 0}$, the conditions: $\phi(k; x) \in \mathbb{X}_0$ and $|\phi(k; x)| \leq |\psi(k; z)| \leq b'\lambda^k |z| \leq b\lambda^k |x|$ hold with $b = b'(1 + c)$. □

Proposition 26. *The admissible set $\mathbb{X}_0$ and restricted feasible set $\bar{\mathcal{X}}_N$ satisfy the following:*

$$\mathbb{X}_0(\bar{V}) \subseteq \bar{\mathcal{X}}_N \text{ for all } \bar{V} \geq 0, \text{ and } \bar{\mathcal{X}}_N \subseteq \bigcup_{\bar{V} \geq 0} \mathbb{X}_0(\bar{V}) \tag{23}$$

Proof. The fact that $\mathbb{X}_0(\bar{V}) \subseteq \bar{\mathcal{X}}_N$ for all $\bar{V} \geq 0$ follows directly from the definitions (22) and the observation that $\mathbb{X}_0$ contains only states that can be steered to $\text{int}(\mathbb{X}_f)$. Hence: $\bar{\mathcal{X}}_N$ is the set of states that can be brought to the interior of $\mathbb{X}_f$ with feasible inputs, and $\mathbb{X}_0(\bar{V})$ is the set that can be brought to the interior of $\mathbb{X}_f$ with feasible inputs *and* cost not exceeding $\bar{V}$.

We next establish the second inclusion. First we show that the sets $\mathbb{X}_0(\bar{V})$ are nested: $\bar{V}_2 \geq \bar{V}_1$ implies $\mathbb{X}_0(\bar{V}_2) \supseteq \mathbb{X}_0(\bar{V}_1)$. Assume an arbitrary $x \in \mathbb{X}_0(\bar{V}_1)$, and corresponding $(x, \mathbf{u}) \in \bar{\mathcal{Z}}_r(\bar{V}_1)$. We show that $x \in \mathbb{X}_0(\bar{V}_2)$. Let $\beta_1 := \bar{V}_1/\alpha$, $\beta_2 := \bar{V}_2/\alpha$ and $x(N) := \phi(N; x, \mathbf{u})$. We have that:

$$V_N^{\beta_2}(x, \mathbf{u}) = V_N^{\beta_1}(x, \mathbf{u}) + (\beta_2 - \beta_1)V_f(x(N))$$

First notice that if $x \in r\mathbb{B}$, $V_N^{\beta_1}(x, \mathbf{u}) \leq \beta_1 V_f(x)$, and this implies that $V_N^{\beta_2}(x, \mathbf{u}) \leq \beta_2 V_f(x)$, so the required inequality is established in $r\mathbb{B}$. Then notice that $V_f(x(N)) = \alpha' < \alpha$, which gives

$$\begin{aligned} V_N^{\beta_2}(x, \mathbf{u}) &\leq \bar{V}_1 + (\bar{V}_2 - \bar{V}_1)(\alpha'/\alpha) = \bar{V}_1(1 - \alpha'/\alpha) + \bar{V}_2(\alpha'/\alpha) \\ &\leq \bar{V}_2(1 - \alpha'/\alpha) + \bar{V}_2(\alpha'/\alpha) = \bar{V}_2 \end{aligned}$$

and we conclude $x \in \mathbb{X}_0(\bar{V}_2)$. Next we establish that for every point $x_0 \in \bar{\mathcal{X}}_N$ there exists a $\bar{V}_0 > 0$ such that $x_0 \in \mathbb{X}_0(\bar{V})$ for all $\bar{V}$ satisfying $\bar{V} \geq \bar{V}_0$. Take an arbitrary $x_0 \in \bar{\mathcal{X}}_N$ and a corresponding $\mathbf{u}_0 \in \mathbb{U}^N$ that satisfies $\phi(N; x_0, \mathbf{u}_0) \in \text{int}(\mathbb{X}_f)$. If $x_0 \in r\mathbb{B}$, add the restriction to $\mathbf{u}_0$ that $V_N(x_0, \mathbf{u}_0) \leq V_f(x_0)$, where $V_N(\cdot)$ continues to denote the original cost function, i.e., with non-inflated terminal cost. Such a $\mathbf{u}_0$ exists because of Proposition 5, which establishes that the optimal input sequence, for example, has this property. And since $\beta \geq 1$, it follows that $V_N^\beta(x_0, \mathbf{u}_0) \leq \beta V_N(x_0, \mathbf{u}_0) \leq \beta V_f(x_0)$, if $x_0 \in r\mathbb{B}$. Then denote by α' the terminal cost $\alpha' = V_f(\phi(N; x_0, \mathbf{u}_0))$, and we have that $\alpha' < \alpha$. Then define $\bar{V}_0 := \left(\frac{1}{1-\alpha'/\alpha}\right) \sum_{i=0}^{N-1} \ell(\phi(i; x_0, \mathbf{u}_0)), u_0(i))$. A direct computation gives $V_N^\beta(x_0, \mathbf{u}_0) = \bar{V}_0$, and, if $x_0 \in r\mathbb{B}$, $V_N^\beta(x_0, \mathbf{u}_0) \leq \beta V_f(x_0)$. Therefore $x_0 \in \mathbb{X}_0(\bar{V}_0)$, and by the nesting property, $x_0 \in \mathbb{X}_0(\bar{V})$ for all $\bar{V}$ satisfying $\bar{V} \geq \bar{V}_0$, and the limit is established. □

Theorem 27. *Under Assumptions 1, 23, 3 and 4, the origin of the closed-loop system* (7) *is SRES on* $\bar{\mathcal{C}}_\rho$.

Proof. (Robust feasibility) Suppose that $x \in \bar{\mathcal{C}}_\rho$ and let $z = (x, \mathbf{u})$ be the corresponding augmented state where $\mathbf{u}$ is a suboptimal sequence computed for the measured state $x_m = x + e$, $e \in \rho\mathbb{B}$. We recall that from (24), it follows that $V_N^\beta(z) \leq \bar{V}$ and that $z_m \in \bar{\mathcal{S}}_\rho \subseteq \bar{\mathcal{Z}}_r$. Moreover, we define $\tilde{z}^+ = (\tilde{x}^+, \tilde{\mathbf{u}})$ where we recall that $\tilde{x}^+ = f(x_m, u(0; x_m))$ is the successor state for the nominal system (no disturbance). The quantity $V_N^\beta(\tilde{z}^+)$ is the cost along the nominal evolution from z_m (no disturbance). Since $V_N^\beta(\cdot)$ is an exponential Lyapunov function for the nominal system on set $\bar{\mathcal{Z}}_r$ (Lemma 24), we have from Proposition 10 that $V_N^\beta(\tilde{z}^+) \leq \bar{\gamma} V_N^\beta(z_m)$ for some $0 < \bar{\gamma} < 1$. Thus, it follows that $V_N^\beta(\tilde{x}^+, \tilde{\mathbf{u}}) \leq \bar{\gamma}\bar{V} < \bar{V}$. Recalling that: $\tilde{x}^+ - x_m^+ = f(x_m, u(0; x_m)) - f(x, u(0; x_m)) - d - e^+$, it follows from continuity of V_N^β that there exists a $\tilde{\delta}_1 > 0$ such that: $V_N^\beta(x_m^+, \tilde{\mathbf{u}}) < \bar{V}$ holds (strictly) for all $(z_m, e, d, e^+) \in \bar{\mathcal{S}}_\rho \times \tilde{\delta}_1\mathbb{B} \times \tilde{\delta}_1\mathbb{B} \times \tilde{\delta}_1\mathbb{B}$. Since $\beta = \max\{1, \bar{V}/\alpha\}$, from (20b) and given that $V_N^\beta(x_m^+, \mathbf{u}^+) = \sum_{k=0}^{N-1} \ell(\phi(k; x_m^+, \mathbf{u}^+)) + \beta V_f(\phi(N; x_m^+, \mathbf{u}^+))$, it follows that $V_f(\phi(N; x_m^+, \mathbf{u}^+)) < \alpha$, which proves that $\phi(N; x_m^+, \mathbf{u}^+) \in \text{int}(\mathbb{X}_f)$. From continuity of V_N^β it also follows that we can choose $\rho > 0$ sufficiently small that $\bar{V}_N^\rho(z_m^+) \leq \bar{V}$. Taking $\delta = \min\{\rho, \tilde{\delta}_1\}$ we have proved that $z_m^+ \in \bar{\mathcal{S}}_\rho$ for all $(z_m, e, d, e^+) \in \bar{\mathcal{S}}_\rho \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}$. This implies:

$$x(k) \in \bar{\mathcal{C}}_\rho \subseteq \mathbb{X}_0 \quad \text{ for all } k \in \mathbb{I}_{\geq 0}$$

and also that $x_m(k) \in \mathbb{X}_0$ for all $k \in \mathbb{I}_{\geq 0}$. Hence, (10a) holds (with $\mathcal{X}_N$ replaced by $\mathbb{X}_0$).

(Robust stability) This part follows exactly the corresponding part in the proof of Theorem 21 and is omitted. (Note that the feasibility recovery step (11) is not mentioned in the robust stability part of the proof of Theorem 21.) □

References

[1] G. Pannocchia, J. B. Rawlings, and S. J. Wright. Inherently robust suboptimal nonlinear MPC: theory and application. Accepted for presentation in 50th IEEE CDC-ECC, 2011.