Model Predictive Control: Current Status and Future Challenges

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Outline
1. Overview
2. MPC at the Small Scale
   - MPC to replace PID
3. MPC at the Large Scale
   - Large, networked systems
4. MPC and State Estimation
5. Conclusions

Separation of the control problem

Input/output description

Estimation problem

Control problem

Regulation problem

State description

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The Regulation Problem

- Model and constraints
- Objective function
- Feedback

Models and constraints

**Linear dynamics and constraints**

\[
\frac{dx}{dt} = Ax + Bu \\
y = Cx \\
Du \leq d \\
Hx \leq h \\
u^2 \\
x^2
\]

Objective Function

**Controller objective function**

\[
\Phi(x, u(t)) = \sum_{k=0}^{\infty} L(x_k, u_k) \\
L(x, u) = x'Qx + u'Ru, \quad \text{quadratic measure common}
\]

Process models (cont.)

**Nonlinear dynamics and constraints**

\[
\frac{dx}{dt} = f(x, u) \\
y = g(x) \\
u \in U \\
x \in X \\
\text{Past} \quad \text{Present} \quad \text{Future}
\]

Objective Function

Controller objective function

\[
\Phi(x, u(t)) = \sum_{k=0}^{\infty} L(x_k, u_k) \\
L(x, u) = x'Qx + u'Ru, \quad \text{quadratic measure common}
\]
One technique for obtaining a feedback controller synthesis from knowledge of open-loop controllers is to measure the current control process state and then compute very rapidly for the open-loop control function. The first portion of this function is then used during a short time interval, after which a new measurement of the process state is made and a new open-loop control function is computed for this new measurement. The procedure is then repeated.

— Lee and Markus (1967)

Foundations of Optimal Control Theory

Everything has been thought of before, but the problem is to think of it again.

— Goethe

A finite horizon objective function may not even stabilize!
• How is this possible?

Adding a terminal constraint ensures stability
• May cause infeasibility
• Open-loop predictions not equal to closed-loop behavior
Infinite horizon solution

- The infinite horizon ensures stability
- Open-loop predictions equal to closed-loop behavior
- May be difficult to implement

\[ \Phi_{k+1} = \Phi_k - L(x_k, u_k) \]

\[ \Phi_{k+2} = \Phi_{k+1} - L(x_{k+1}, u_{k+1}) \]

Full Enumeration

- Unconstrained solution: LQ regulator (?)
  \[ u = Kx \]
- Constrained solution: MPC
  \[ u_0 = K_i x + b_i \]
  in which \( i \) enumerates different possible active sets for the inequality constraints (?)
- There are \( 3^{mN} \) different active sets

\[ \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_k \end{bmatrix} \leq \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \]
\[ k = 1, \ldots, N \]

The active set table

<table>
<thead>
<tr>
<th>( i )</th>
<th>constraint set</th>
<th>( K_i )</th>
<th>( b_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( { \pi, \pi } )</td>
<td>0</td>
<td>( \pi )</td>
</tr>
<tr>
<td>2</td>
<td>( { \pi, -} )</td>
<td>0</td>
<td>( \pi )</td>
</tr>
<tr>
<td>3</td>
<td>( { \pi, u } )</td>
<td>0</td>
<td>( \pi )</td>
</tr>
<tr>
<td>4</td>
<td>( { -, \pi } )</td>
<td>( K_4 )</td>
<td>( b_4 )</td>
</tr>
<tr>
<td>5</td>
<td>( { -, -, } )</td>
<td>( K_6 )</td>
<td>( b_6 )</td>
</tr>
<tr>
<td>6</td>
<td>( { u, -} )</td>
<td>( K_6 )</td>
<td>( b_6 )</td>
</tr>
<tr>
<td>7</td>
<td>( { u, \pi } )</td>
<td>0</td>
<td>( u )</td>
</tr>
<tr>
<td>8</td>
<td>( { u, -} )</td>
<td>0</td>
<td>( u )</td>
</tr>
<tr>
<td>9</td>
<td>( { u, u } )</td>
<td>0</td>
<td>( u )</td>
</tr>
</tbody>
</table>

Example 1 — First order plus time delay

\[ u_0 = K_i x + b_i \]

\[ N = 2 \]

\[ N = 4 \]

- The first example is a first order plus time delay (FOPTD) system (?)
  \[ G_1(s) = \frac{e^{-2s}}{10s + 1} \]
  sampled with \( T_s = 0.25 \)
- The input is assumed to be constrained \( |u| \leq 1.5 \)
- The control horizon is \( N = 4 \)
Setpoint change and load disturbances

- In all simulations the setpoint is changed from 0 to 1 at time zero.
- At time 25 a load disturbance passing through the same dynamics as the plant of magnitude $-0.25$ enters the system.
- At time 50 the disturbance magnitude becomes $-1$ (which makes the setpoint 1 unreachable).
- Finally at time 75 the disturbance magnitude becomes $-0.25$ again.

FOPTD system: nominal case

FOPTD system: noisy case

FOPTD system: effect of plant/model mismatch.
Computation time for (complete) enumeration

- The computational burden of CLQ is comparable to that of PID.
- The CPU is a 1.7 GHz Athlon PC running Octave

<table>
<thead>
<tr>
<th>Average CPU time (ms)</th>
<th>Maximum CPU time (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PID</td>
<td>0.05</td>
</tr>
<tr>
<td>CLQ</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Electrical power distribution

MPC at the Large Scale

- Most large-scale systems consist of networks of interconnected/interacting subsystems
  - Chemical plants, electrical power grids, water distribution networks, . . .

- Traditional approach: Decentralized control
  - Wealth of literature from the early 1970’s on improved decentralized control (???)
  - Well-known that poor performance may result if the interconnections are not negligible
MPC at the Large Scale

- Steady increase in available computational power has provided the opportunity for centralized control
- Most practitioners view centralized control of large, networked systems as impractical and unrealistic
  - Centralized control law grows exponentially with system size
  - Difficult to tailor a centralized controller to meet operational objectives
- A divide and conquer strategy is essential for control of large, networked systems
- Centralized control: A benchmark control framework for comparing and assessing other control formulations

Nomenclature: Consider Two Interacting Units

<table>
<thead>
<tr>
<th>Objective functions</th>
<th>( \Phi_1(u_1, u_2), \Phi_2(u_1, u_2) ) and ( \Phi(u_1, u_2) = w_1 \Phi_1(u_1, u_2) + w_2 \Phi_2(u_1, u_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decision variables for units</td>
<td>( u_1 \in \Omega_1, \ u_2 \in \Omega_2 )</td>
</tr>
<tr>
<td>Decentralized Control</td>
<td>( \min_{u_1 \in \Omega_1} \Phi_1(u_1), \min_{u_2 \in \Omega_2} \Phi_2(u_2) ) (Nash equilibrium)</td>
</tr>
<tr>
<td>Communication-based Control</td>
<td>( \min_{u_1 \in \Omega_1} \Phi_1(u_1, u_2), \min_{u_2 \in \Omega_2} \Phi_2(u_1, u_2) ) (Pareto optimal)</td>
</tr>
<tr>
<td>Cooperation-based Control</td>
<td>( \min_{u_1 \in \Omega_1} \Phi(u_1, u_2), \min_{u_2 \in \Omega_2} \Phi(u_1, u_2) ) (Pareto optimal)</td>
</tr>
<tr>
<td>Centralized Control</td>
<td>( \min_{u_1, u_2 \in \Omega_1 \times \Omega_2} \Phi(u_1, u_2) )</td>
</tr>
</tbody>
</table>

Noninteracting systems

![Noninteracting systems graph](image)

Weakly interacting systems

![Weakly interacting systems graph](image)
Moderately interacting systems

Strongly interacting (conflicting) systems

Strongly interacting (conflicting) systems

Application chemical plant
The conditional density function

For the linear, time invariant model with Gaussian noise,

\[
x(k + 1) = Ax + Bu + Gw \\
y = Cx + v
\]

\[w \sim N(0, Q) \quad v \sim N(0, R) \quad x(0) \sim N(\bar{x}_0, Q_0)\]

We can compute the conditional density function exactly

\[
p_{x|y}(x|y(k-1)) = N(\hat{x}^-, P^-) \quad \text{(before } y(k)\text{)}
\]

\[
p_{x|y}(x|y(k)) = N(\hat{x}, P) \quad \text{(after } y(k)\text{)}
\]

Large \(R\), ignore the measurement, trust the forecast

Medium \(R\), blend the measurement and the forecast
Small $R$, trust the measurement, override the forecast

Large $R$, $y$ measures $x_1$ only

Medium $R$, $y$ measures $x_1$ only

Small $R$, $y$ measures $x_1$ only
### The challenge of nonlinear estimation

**Linear Estimation**

- **Estimation Possibilities:**
  1. One state is the optimal estimate
  2. Infinitely many states are optimal estimates (unobservable)

**Nonlinear Estimation**

- **Estimation Possibilities:**
  1. One state is the optimal estimate
  2. Infinitely many states are optimal estimates (unobservable)
  3. Finitely many states are locally optimal estimates

---

### Full information estimate of trajectory

The trajectory of states

\[ X(T) := \{ x(0), \ldots, x(T) \} \]

Maximizing the conditional density function

\[
\max_{X(T)} p_{X|Y}(X(T) | Y(T))
\]

---

### Equivalent optimization problem

Using the model and taking logarithms

\[
\min_{X(T)} V_0(x_0) + \sum_{j=1}^{T-1} L_w(w_j) + \sum_{j=0}^{T} L_v(y_j - h(x_j))
\]

subject to \( x(j+1) = F(x, u) + w \)

\[ V_0(x) := -\log(p_0(x)) \]
\[ L_w(w) := -\log(p_w(w)) \]
\[ L_v(v) := -\log(p_v(v)) \]

---

### Arrival cost and moving horizon estimation

Most recent \( N \) states \( X(T - N : T) := \{ x_{T-N}, \ldots, x_T \} \)

Optimization problem

\[
\min_{X(T-N:T)} V_{T-N}(x_{T-N}) + \sum_{j=T-N}^{T-1} L_w(w_j) + \sum_{j=T-N}^{T} L_v(y_j - h(x_j))
\]

subject to \( x(j+1) = F(x, u) + w \).
Arrival cost approximation

The statistically correct choice for the arrival cost is the conditional density of \( x_{T-N} | Y(T-N-1) \)

\[
V_{T-N}(x) = - \log p_{x_{T-N}}(x | Y(T-N-1))
\]

Arrival cost approximations (?)
- uniform prior (and large \( N \))
- EKF covariance formula
- MHE smoothing

Sequential Monte Carlo Sampling

- Represent distribution at time \( k \) via \( N \) samples (or particles), and weights,
  - Particles, \( x^i(T), i = 1, \ldots, N \)
  - Weights, \( q^i(T), i = 1, \ldots, N \)
- Any moment can be approximated as,
  - \( E(f(x(T))) \approx \sum_i^N q^i(T)f(x^i(T)) \)
- Point estimate, Highest Posterior Density regions, etc. may be computed from \( \{x^i(T), q^i(T)\} \)
- Capturing system dynamics and measurements requires efficient algorithm for propagating particles and weights over time,
  - \( \{x^i(T-1), q^i(T-1)\} \longrightarrow \{x^i(T), q^i(T)\} \)
- Combine Sequential Monte Carlo Sampling with Importance Sampling

Particle Filtering Methodology

- A convenient importance function is 
  \[
  \pi(x^i(T)|x^i(T-1), y(T)) = p(x(T)|x^i(T-1))
  \]

Conclusions

- MPC is finding new application on small-scale, fast loops as well as large-scale, networked systems.
- State estimation is an integral component of MPC and remains a current research challenge.
- MHE and particle filtering are high-quality solutions for nonlinear models. They require more user experience to set up properly and more computational resources to execute.
  The payoff can be substantial, however.
Future Challenges

- MPC of large-scale systems
  - Develop identification methods for “minimal” modeling of the unit interactions.
  - Exciting applications in many fields!
- State estimation in MPC
  - Process systems are typically unobservable or ill-conditioned, i.e. nearby measurements do not imply nearby states.
  - We must decide on the subset of states to reconstruct from the data – an additional part to the modeling question.
  - Nonlinear systems produce multi-modal densities. We need better solutions for handling these multi-modal densities in real time.

Further Reading I

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