Using Moving Horizon Estimation to Overcome Extended Kalman Filtering Failure

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Outline

• State estimation overview
  – problem formulation
  – extended Kalman filter
  – moving horizon estimation

• Effect of arrival cost

• Closed-loop control and plant-model mismatch

• Conclusions
State Estimation Overview

Unmeasured Disturbances

Process
\[ x_{k+1} = F(x_k, u_k, w_k) \]

Sensor Noise
\[ v_k \]

Sensor
\[ y_k = h(x_k) + v_k \]

Measurements
\[ y_k \]

Estimator

State Estimate
\[ \hat{x}_k \]
What estimate is desired?

- We will consider the model

\[
x_{k+1} = F(x_k, u_k) + Gw_k
\]
\[
y_k = h(x_k) + v_k
\]

- If we knew the a posteriori distribution

\[
p(x_T|y_0, \ldots, y_T)
\]

what point estimate should we calculate?
What estimate is desired?

For unconstrained linear estimation with Gaussian noise, the mean and mode of the probability distribution are the same.

Optimal estimator: Kalman filter (recursive).
What estimate is desired?

The mean and mode of the probability distribution are generally different.

We would like to solve for the the maximum a posteriori estimate (MAP), i.e. the mode of this distribution.
Extended Kalman Filtering

- Approximates

\[ \hat{x}_{T|T} \approx \arg \max_{x_T} p(x_T|y_0, \ldots, y_T) \]

- Extension of the Kalman filter to nonlinear systems via linearization

- Summarizes past data with the covariance matrix

- Computationally trivial

- Most popular industrial method
Moving Horizon Estimation

- Approximates
  \[ \{ \hat{x}_{T-N|T}, \ldots, \hat{x}_T | T \} \approx \arg \max_{x_{T-N}, \ldots, x_T} p(x_{T-N}, \ldots, x_T | y_0, \ldots, y_T) \]

- Accurately employs the non-linear model

- Can incorporate constraints

- Requires on-line optimization

- Arrival cost?
Effect of Arrival Cost: An Illustrative Example

- Well-mixed, gas phase, batch reactor
- Estimate the partial pressures of A and B
- Model
  \[
  \frac{dx}{dt} = \begin{bmatrix} -2 & 1 \end{bmatrix}^T k P_A^2
  \]
- Measure the total pressure
  \[
  \gamma = P_A + P_B
  \]
- Poor initial guess
  \[
  x_o = \begin{bmatrix} 3 & 1 \end{bmatrix}^T \text{ vs. } \tilde{x}_o = \begin{bmatrix} 0.1 & 4.5 \end{bmatrix}^T
  \]
Estimator Comparisons

Actual state (red)
Estimated state (blue)

$P_A \geq 0, P_B \geq 0$
Horizon length of 2 minutes
Maximum a Posteriori Distribution

\[ p(x_T|y_0, \ldots, y_T) \]

Maximum a posteriori distribution exhibits multiple optima.
Extended Kalman Filtering and Moving Horizon Estimation

Extended Kalman Filtering
\[ \approx p(x_T | y_0, \ldots, y_T) \]

Moving Horizon Estimation
\[ \approx \max_{x_{T-N}, \ldots, x_{T-1}} p(x_{T-N}, \ldots, x_T | y_0, \ldots, y_T) \]

 Depends on the arrival cost

- approximate the process as a time-varying linear system?
- uniform prior?
Arrival Cost Strategies

Arrival cost approximations: smoothing update (assumes process is a time-varying linear system) or uniform prior

**Smoothing Update**

$$\max_{x_1, \ldots, x_3} p(x_1, \ldots, x_4 | y_0, \ldots, y_4)$$

**Uniform Prior**

$$\max_{x_1, \ldots, x_3} p(x_1, \ldots, x_4 | y_0, \ldots, y_4)$$

The prior (arrival cost) dominates and distorts the information contained in the data. Better to use a uniform prior if global optimization possible.
Arrival Cost Strategies: Effect of Horizon Length

Arrival cost approximations: smoothing update (assumes process is a time-varying linear system) or uniform prior

**Smoothing Update**

\[
\max_{x_1, \ldots, x_9} p(x_1, \ldots, x_{10} \mid y_0, \ldots, y_{10})
\]

**Uniform Prior**

\[
\max_{x_1, \ldots, x_9} p(x_1, \ldots, x_{10} \mid y_0, \ldots, y_{10})
\]

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Arrival Cost Conclusions

- The EKF can predict only one optimum. Estimation behavior dictated by initial region of attraction.

- Behavior of MHE depends upon the arrival cost.
  - Approximating the past behavior as a time-varying, linear system introduces significant bias
  - Longer horizon can overcome effects of poor arrival cost
  - Best (current) option: uniform prior with global optimization
Closed-Loop Control

We will use the NMPC toolbox for the estimator, regulator, and target calculation (local optimization) \[3\].
Disturbance models for nonlinear models

- For offset free control, must account for discrepancies between the plant and the model.

- Augment the state with a disturbance model:

\[
\begin{align*}
    x_{k+1} &= F(x_k, u_k + X_u d_k) + Gw_k \\
    y_k &= h(x_k) + X_y d_k + v_k \\
    d_{k+1} &= d_k + \xi_k \\
    \xi_k &\sim \mathcal{N}(0, Q_d)
\end{align*}
\]

Implies that \( d_k \) is stochastic!
Plant-model mismatch: exothermic CSTR example

\[ x = \begin{bmatrix} c_A \\ T \end{bmatrix} \]

\[ u = T_c \]

\[ |\Delta u_k| \leq 15 \text{ K} \]

\[ y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k + d_k \]

Small mismatch in activation energy between the plant and the model.
Output disturbance model generates multiple steady states!

Multiple optima arise in the estimator. Do these optima affect control performance?

<table>
<thead>
<tr>
<th>$c_A$ (mol/l)</th>
<th>Output $T$ (K)</th>
<th>Disturbance $d$ (K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.851</td>
<td>326.2</td>
</tr>
<tr>
<td>2</td>
<td>0.583</td>
<td>344.4</td>
</tr>
<tr>
<td>3</td>
<td>0.177</td>
<td>371.8</td>
</tr>
</tbody>
</table>

Model Steady States for a Plant with $T_c = 300$ K, $T = 350$ K

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Example: Disturbance rejection

• Consider a disturbance in the feed:

![Graph showing temperature and concentration changes over time]

• Estimators:
  1. EKF
  2. MHE with $N = 2$, smoothing update
  3. MHE with $N = 10$, no initial penalty
  4. MHE with $N = 10$, constant initial penalty
Exothermic CSTR Results

Graph showing the output temperature (T) and input temperature (Tc) over time for different models:
- EKF
- MHE, N=2
- MHE, N=10
- uMHE, N=10
- cMHE, N=10

Time [hr] ranges from 0 to 5.

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Comparison to Linear MPC Results

### Output T [K]

- **set point**
- **LMPC**
- **cMHE, N=10**

### Input Tc [K]

- **LMPC**
- **cMHE, N=10**

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Plant-model mismatch: maximum yield example

\[ x = \begin{bmatrix} c_A & c_B \end{bmatrix}^T \]

\[ u = \begin{bmatrix} T_c & c_{Af} \end{bmatrix}^T \]

\[ u_k = \begin{bmatrix} T_c \\
 0 
\end{bmatrix} + \begin{bmatrix} 1 \\
 0 
\end{bmatrix} d_k \]

\[ y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k \]
Maximum Yield CSTR Results

Output Disturbance $d_k$ [mol/l]

Output $C_b$ [mol/l]

Set point

MHE

Target calculation fails

EKF

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Maximum Yield CSTR Results

- **c_A [mol/l]**
  - EKF
  - MHE
  - Set point
  - Target calculation fails

- **Disturbance [l/hr]**
  - EKF
  - MHE
  - Target calculation fails
Conclusions

- Integrated disturbance models can induce multiple optima with nonlinear models.

- EKF: poor state tracking results in poor control.

- MHE
  1. better control than the EKF
  2. increased horizon length results in improved performance

- No significant improvement in nonlinear over linear control for disturbance rejection.

- If you want better disturbance rejection, you need a better disturbance model.
Acknowledgements

- Organizers of the Gordon Research Conference for Statistics in Chemistry & Chemical Engineering

- Wen-shiang Chen and Prof. Bhavik Bakshi (Ohio State University)
Questions?
Improving MHE

• Primary concern: longer horizon length = greater computational expense

• Basis of MHE:

$$\max_{x_{T-N+1}, \ldots, x_T} p(x_{T-N+1}, \ldots, x_T | y_0, \ldots, y_T)$$

$$= \max_{x_{T-N+1}, \ldots, x_T} p(x_{T-N+1} | y_0, \ldots, y_T) \left( \prod_{k=T-N+1}^{T-1} p(x_{k+1} | x_k) \right) \left( \prod_{k=T-N+1}^{T} p(y_k | x_k) \right)$$

Arrival Cost

Estimation Horizon

• If we had a better estimate for $p(x_{T-N+1} | y_0, \ldots, y_T)$, we could shorten the estimation horizon

• Can we use Monte Carlo filters to estimate this density?
State Estimation via Monte Carlo Filters

- Basic idea: reconstruct state estimates from simulations of the stochastic process

\[ \int h(x)P(x)dx = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} h(x^i) \]

- Most MC filters propose estimation of the mean

\[ E[x] = \int xP(x)dx \approx \frac{1}{N} \sum_{i=1}^{N} x^i \]

- Permits use of any combination of model, random noise

- We will consider rejection sampling [1]
• MHE estimates the mode

• Monte Carlo filters must estimate the entire probability density to calculate the mode
  1. density estimation
  2. optimization
Example: density estimation of a normal distribution

Draw samples from the underlying distribution.

Apply a symmetric “kernel” density at each sample.

Sum the kernel densities to approximate the underlying distribution.
A Simple Example

- Well-mixed, gas phase, batch reactor
- Estimate the partial pressures of A and B
- Model
  \[
  \frac{dx}{dt} = \begin{bmatrix} -2 & 1 \end{bmatrix}^T k P_A^2
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  \]
A Posteriori Comparison: Actual vs. Monte Carlo $p(x_1|y_0, y_1)$

Actual

Monte Carlo Reconstruction (100 Accepted Samples)
Comments on Monte Carlo Estimation

• Not very accurate estimation of the mode.

Sources of error:

1. finite number of samples
2. density estimation approximation

• Simple to code

• May provide a useful estimate of the arrival cost if computationally inexpensive
Density estimation: the curse of dimensionality

<table>
<thead>
<tr>
<th>Dimensionality</th>
<th>Required Sample Size</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>19</td>
</tr>
<tr>
<td>3</td>
<td>67</td>
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<td>43700</td>
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<tr>
<td>9</td>
<td>187000</td>
</tr>
<tr>
<td>10</td>
<td>842000</td>
</tr>
</tbody>
</table>

Sample size required to ensure that the relative mean square error at zero (a single point) is less than 0.1. The underlying distribution is a standard multivariate normal density.
References

