Outline

- Nonlinear Dynamic Systems
- Extended Kalman Filter (EKF)
  - Simulation Example
- Unscented Kalman Filter (UKF)
- Conclusions
- Further Reading
Nonlinear Dynamic Systems

- For the nonlinear model with Gaussian noise

\[
\begin{align*}
x(k+1) &= F(x, u) + G(x, u)w \\
y(k) &= h(x) + v
\end{align*}
\]

\[
\begin{align*}
w &\sim N(0, Q) & v &\sim N(0, R) & x(0) &\sim N(\bar{x}_0, Q_0)
\end{align*}
\]

- Consider the linearization at every \( k \)

\[
\begin{align*}
\bar{A}(k) &= \left. \frac{\partial F(x, u)}{\partial x} \right|_{\bar{x}(k), u(k)} \\
\bar{C}(k) &= \left. \frac{\partial h(x)}{\partial x} \right|_{\bar{x}(k), u(k)} \\
\bar{G}(k) &= G(\bar{x}(k), u(k))
\end{align*}
\]

Extended Kalman Filter (EKF) Recursion

- Forecast

\[
\begin{align*}
\hat{x}^-(k+1) &= F(\hat{x}, u) \\
P^-(k+1) &= \bar{A}(k)P(k)\bar{A}'(k) + \bar{C}(k)Q\bar{C}'(k) \\
\hat{x}^-(0) &= \bar{x}_0 & P^-(0) &= Q_0
\end{align*}
\]

- Correction

\[
\begin{align*}
\hat{x}(k) &= \hat{x}^-(k) + L(k)(y(k) - h(\hat{x}^-(k))) \\
L(k) &= P^-(k)\bar{C}'(k)(\bar{C}(k)P^-(k)\bar{C}'(k) + R)^{-1} \\
P(k) &= P^-(k) - L(k)\bar{C}(k)P^-(k)
\end{align*}
\]
Extended Kalman Filter — Remarks

- EKF has a similar recursion in structure to the KF with
  - Mean propagation through the full nonlinear model
  - Covariance propagation through the linearized model

- Resulting error from linearization may cause filter divergence

EKF on Batch Reactor Example

- Estimate the concentrations of A, B, and C
- Model

\[
\begin{bmatrix}
  \frac{d}{dt} c_A \\
  \frac{d}{dt} c_B \\
  \frac{d}{dt} c_C 
\end{bmatrix} =
\begin{bmatrix}
  -1 & 0 & k_1 c_A - k_{-1} c_B c_C \\
  1 & -2 & k_2 c_B - k_{-2} c_C \\
  1 & 1 & \text{[Linearized terms]}
\end{bmatrix}
\]

- Measure the total pressure

\[ y = RT (c_A + c_B + c_C) \]

- Poor initial guess

\[ x_0 = \begin{bmatrix} 0.5 & 0.05 & 0 \end{bmatrix}^T \text{ vs. } \bar{x}_0 = \begin{bmatrix} 0 & 0 & 4 \end{bmatrix}^T \]
EKF Results

<table>
<thead>
<tr>
<th>Component</th>
<th>Predicted EKF</th>
<th>Actual Steady-State</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-0.027</td>
<td>0.012</td>
</tr>
<tr>
<td>B</td>
<td>-0.238</td>
<td>0.184</td>
</tr>
<tr>
<td>C</td>
<td>1.137</td>
<td>0.675</td>
</tr>
</tbody>
</table>

Clipped EKF Results

Clipping of States: $c_j < 0 \rightarrow c_j = 0$, $j = A, B, C$
Constrained MHE Results

State Constraints: \( c_j \geq 0, \ j = A, B, C \)

Extended Kalman Filter — Assessment

*The extended Kalman filter is probably the most widely used estimation algorithm for nonlinear systems.*

*However, more than 35 years of experience in the estimation community has shown that it is difficult to implement, difficult to tune, and only reliable for systems that are almost linear on the time scale of the updates.*

*Many of these difficulties arise from its use of linearization.*

Unscented Kalman Filter (UKF)

- Given $\hat{x}$ and $P$, choose sample points, $z^i$, and weights, $w^i$, such that
  $$\hat{x} = \sum_i w^i z^i \quad P = \sum_i w^i (z^i - \hat{x})(z^i - \hat{x})'$$

- Similarly, given $w \sim N(0, Q)$ and $v \sim N(0, R)$, choose sample points $n^i$ for $w$ and $m^i$ for $v$.

UKF Prediction Step

- Propagate sigma points with the nonlinear model
  $$z^i_{k+1} = F(z^i_k, u_k) + G(z^i_k, u_k) n^i_k$$

- From these compute the forecast
  $$\hat{x}^-_{k+1} = \sum_i w^i z^i_{k+1} \quad P^-_{k+1} = \sum_i w^i (z^i_{k+1} - \hat{x}^-_{k+1})(z^i_{k+1} - \hat{x}^-_{k+1})'$$
UKF Measurement Update

- Measurement forecast:
  \[ \eta_{k+1}^i = h(z_{k+1}^i) + m_i, \quad \tilde{y}_{k+1} = \sum_i w_i \eta_{k+1}^i \]

- Output error: \( \mathcal{Y} := y - \hat{y} \)

UKF Recursion

- First rewrite the Kalman filter update
  \[
  \hat{x} = \hat{x}^+ + L(y - \hat{y}^-) \\
  L = \mathcal{E}((x - \hat{x}^-)\mathcal{Y}') \mathcal{E}(\mathcal{Y}\mathcal{Y}')^{-1} \\
  \mathcal{E}((x - \hat{x}^-)\mathcal{Y}')^{P-C'} \\
  P = P^- - L \mathcal{E}((x - \hat{x}^-)\mathcal{Y}')^{CP^-}\]

- Approximate the two expectations with the sigma point samples
  \[
  \mathcal{E}((x - \hat{x}^-)\mathcal{Y}') \approx \sum_i w_i (z^i - \hat{x}^-)(\eta^i - \hat{y}^-)' \\
  \mathcal{E}(\mathcal{Y}\mathcal{Y}') \approx \sum_i w_i (\eta^i - \hat{y}^-)(\eta^i - \hat{y}^-)' 
  \]
UKF Assessment

- Does not linearize at a single point. Samples the nonlinearity at several places ($2n$).

- Computationally efficient.

- Does not require even the Jacobian $\frac{\partial F(x, u)}{\partial x}$ of the model.

- Has been tested on simulation examples, including process control examples (exothermic CSTR, pH). (Romanenko and Castro, 2004; Romanenko et al., 2004).

UKF Assessment (cont’d)

- Attractive alternative if the EKF gives convergence problems or proves difficult to tune.

- Recently published work incorporates constraints in the UKF formulation (Vachhani et al., 2006)
  - Performance not yet compared to other nonlinear and constrained approaches such as
    - Moving Horizon Estimation (optimization based)
    - Particle Filtering (sampling based)
Conclusions

Here we have learned ...

- State estimation approaches for unconstrained nonlinear systems
  - Extended Kalman Filter
    - Example showed EKF divergence
  - Unscented Kalman Filter
    - Attractive alternative if the EKF fails
    - Incorporation of constraints is under development

Further Reading


