Robustness of Suboptimal Model Predictive Control

James B. Rawlings

Department of Chemical and Biological Engineering
University of Wisconsin–Madison

Insitut für Systemtheorie und Regelungstechnik
Universität Stuttgart
Stuttgart, Germany
June 24, 2011
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Why study robustness of *suboptimal* MPC?

- *Cooperative*, distributed MPC is a special case of *suboptimal* MPC. Anything we establish about suboptimal MPC can be applied to cooperative, distributed MPC (and optimal MPC!)
- Suboptimal MPC has an interesting feature: a nonunique, point-to-set control law $u \in \kappa_N(x)$.
- *Optimal* solution of nonconvex

$$
\mathbb{P}_N(x) : \min_{u \in U_N} V_N(x, u)
$$

cannot be computed online for *any* nonlinear model. Practitioners implement only suboptimal MPC.
- We should know something about its inherent robustness properties.\(^1\)

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\(^1\)Pannocchia, Rawlings, and Wright (2011)
For suboptimal MPC; again, the basic MPC setup

- The system model
  \[ x^+ = f(x, u) \]  
  (1)

- State and input constraints
  \[ x(k) \in X, \quad u(k) \in U \quad \text{for all } k \in \mathbb{I}_{\geq 0} \]

- Terminal constraint (and penalty)
  \[ \phi(N; x, u) \in X_f \subseteq X \]
Feasible sets

- The set of feasible initial states and associated control sequences

\[ \mathcal{Z}_N = \{(x, u) \mid u(k) \in \mathbb{U}, \phi(k; x, u) \in \mathbb{X} \text{ for all } k \in \mathbb{I}_{0:N-1}, \quad \text{ and } \phi(N; x, u) \in \mathbb{X}_f\} \]

and \( \mathbb{X}_f \subseteq \mathbb{X} \) is the feasible terminal set.

- The set of feasible initial states is

\[ \mathcal{X}_N = \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{U}^N \text{ such that } (x, u) \in \mathcal{Z}_N\} \quad (2) \]

- For each \( x \in \mathcal{X}_N \), the corresponding set of feasible input sequences is

\[ \mathcal{U}_N(x) = \{u \mid (x, u) \in \mathcal{Z}_N\} \]
For any state $x \in \mathbb{R}^n$ and input sequence $u \in \mathbb{U}^N$, we define

$$V_N(x, u) = \sum_{k=0}^{N-1} \ell(\phi(k; x, u), u(k)) + V_f(\phi(N; x, u))$$

- $\ell(x, u)$ is the stage cost; $V_f(x(N))$ is the terminal cost
- Consider the finite horizon optimal control problem

$$\mathbb{P}_N(x) : \min_{u \in \mathbb{U}_N} V_N(x, u)$$
Rather then solving $\mathbb{P}_N(x)$ exactly, we consider using any (unspecified) suboptimal algorithm having the following properties.

Let $u \in U_N(x)$ denote the (suboptimal) control sequence for the initial state $x$, and let $\tilde{u}$ denote a *warm start* for the successor initial state $x^+ = f(x, u(0; x))$, obtained from $(x, u)$ by

$$
\tilde{u} := \{u(1; x), u(2; x), \ldots, u(N - 1; x), u_+\}
$$

(3)

$u_+ \in U$ is any input that satisfies the invariance conditions of Assumption 3 for $x = \phi(N; x, u) \in X_f$, i.e., $u_+ \in \kappa_f(\phi(N; x, u))$. 
Suboptimal MPC

- The warm start satisfies $\tilde{u} \in \mathcal{U}_N(x^+)$. 
- The suboptimal input sequence for any given $x^+ \in \mathcal{X}_N$ is defined as any $u^+ \in \mathcal{U}^N$ that satisfies:

\begin{align*}
   u^+ & \in \mathcal{U}_N(x^+) \quad (4a) \\
   V_N(x^+, u^+) & \leq V_N(x^+, \tilde{u}) \quad (4b) \\
   V_N(x^+, u^+) & \leq V_f(x^+) \quad \text{when } x^+ \in r\mathbb{B} \quad (4c)
\end{align*}

in which $r$ is a positive scalar sufficiently small that $r\mathbb{B} \subseteq \mathcal{X}_f$.

- Notice that constraint (4c) is required to hold only if $x^+ \in r\mathbb{B}$, and it implies that $|u^+| \to 0$ as $|x^+| \to 0$.

- Condition (4b) ensures that the computed suboptimal cost is no larger than that of the warm start.
Any $u^0(x^+)$, *optimal* solution to $P_N(x^+)$, satisfies conditions (4a), (4b) for all $x^+ \in X_N$. Moreover, the inequality in condition (4c) is satisfied by $u^0(x^+)$ for all $x^+ \in X_f$.

Therefore, for any $x^+ \in X_N$, there exists a $u^+ \in U_N(x^+)$ satisfying all conditions (4) for all $u \in U_N(x^+)$.

We now observe that $u^+$ is a set-valued map of the state $x^+$, and so is the associated first component $u(0; x^+)$. If we, again, denote the latter map as $\kappa_N(\cdot)$, we can write the evolution of the closed-loop system as the following difference inclusion:

$$x^+ \in \{f(x, u) \mid u \in \kappa_N(x)\}$$

(5)
Inherent robustness of the suboptimal controller

- Consider a process disturbance $d$, $x^+ = f(x, \kappa(x)) + d$
- A measurement disturbance $x_m = x + e$
- Nominal controller with disturbance

\[
\begin{align*}
x^+ & \in f(x, \kappa_N(x_m)) + d \\
x^+ & \in f(x, \kappa_N(x + e)) + d \\
x^+ & \in F_{ed}(x) \quad (6)
\end{align*}
\]

Robust stability; is the system $x^+ \in F_{ed}(x)$ input-to-state stable considering $(d, e)$ as the input.
Assumption 1 (Continuity of system and cost)

The functions $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are continuous, $f(0, 0) = 0$, $\ell(0, 0) = 0$, and $V_f(0) = 0$.

Assumption 2 (Properties of constraint sets)

1. The set $U$ is compact and contains the origin. The sets $X$ and $X_f$ are closed and contain the origin in their interiors, $X_f \subseteq X$.

2. The set $U$ is compact and contains the origin. The sets $X = \mathbb{R}^n$ and $X_f = \text{lev}_{\alpha} V_f = \{ x \in \mathbb{R}^n \mid V_f(x) \leq \alpha \}$, with $\alpha > 0$. 
Basic MPC assumptions

Assumption 3 (Basic stability assumption)

For any $x \in \mathbb{X}_f$ there exists $u := \kappa_f(x) \in \mathbb{U}$ such that $f(x, u) \in \mathbb{X}_f$ and $V_f(f(x, u)) + \ell(x, u) \leq V_f(x)$. 
**Basic MPC assumptions**

**Assumption 3 (Basic stability assumption)**

For any $x \in X_f$ there exists $u := \kappa_f(x) \in U$ such that $f(x, u) \in X_f$ and $V_f(f(x, u)) + \ell(x, u) \leq V_f(x)$.

**Assumption 4**

There exist positive constants $a, a_1', a_2', a_f$ and $\bar{r}$, such that the cost function satisfies the inequalities

\[
\ell(x, u) \geq a_1'|(x, u)|^a 
\]

for all $(x, u) \in X \times U$

\[
V_N(x, u) \leq a_2'|(x, u)|^a 
\]

for all $(x, u) \in \bar{r}B$

\[
V_f(x) \leq a_f|x|^a 
\]

for all $x \in X$
Definition 5 (Exponential stability)

The origin of the difference inclusion $z^+ \in H(z)$ is exponentially stable (ES) on $\mathcal{Z}$, $0 \in \mathcal{Z}$, if there exist scalars $b > 0$ and $0 < \lambda < 1$, such that for any $z \in \mathcal{Z}$, all solutions $\psi(k; z)$ satisfy:

$$\psi(k; z) \in \mathcal{Z}, \quad |\psi(k; z)| \leq b\lambda^k|z| \quad \text{for all } k \in \mathbb{I}_{\geq 0}$$
Definition 6 (Exponential Lyapunov function)

$V$ is an exponential Lyapunov function on the set $\mathcal{Z}$ for the difference inclusion $z^+ \in H(z)$ if there exist positive scalars $a, a_1, a_2, a_3$ such that the following holds for all $z \in \mathcal{Z}$

$$a_1|z|^a \leq V(z) \leq a_2|z|^a, \quad \max_{z^+ \in H(z)} V(z^+) \leq V(z) - a_3|z|^a$$
Nominal exponential stability

Theorem 7 (ES)

Under Assumptions 1, 2.1, 3, and 4, the origin of the closed-loop system

\[ x^+ \in F(x) = \{ f(x, u) \mid u \in \kappa_N(x) \} \]

is ES on (arbitrarily large) compact subsets of \( \mathcal{X}_N \).
Definition 8 (RES)

The origin of the closed-loop system (6) is robustly exponentially stable (RES) on int(\(\mathcal{X}_N\)) if there exist scalars \(b > 0\) and \(0 < \lambda < 1\) such that for all compact sets \(C \subset \mathcal{X}_N\), with \(0 \in \text{int}(C)\), the following property holds:

Given any \(\epsilon > 0\), there exists \(\delta > 0\) such that for all sequences \(\{d(k)\}\) and \(\{e(k)\}\) with \(x(0) = x \in C\) satisfying:

\[
\max_{k \geq 0} |d(k)| \leq \delta, \quad \max_{k \geq 0} |e(k)| \leq \delta
\]

\[x_m(k) = x(k) + e(k) \in \mathcal{X}_N, \quad x(k) \in \mathcal{X}_N, \quad \text{for all } k \in \mathbb{I}_{\geq 0},\]

it follows that

\[|\phi_{ed}(k; x)| \leq b\lambda^k |x| + \epsilon, \quad \text{for all } k \in \mathbb{I}_{\geq 0}. \tag{8}\]
Robust exponential stability of suboptimal MPC

**Definition 9 (SRES)**

The origin of the closed-loop system (6) is *strongly robustly exponentially stable* (SRES) on a compact set $C \subset \mathcal{X}_N$, $0 \in \text{int}(C)$, if there exist scalars $b > 0$ and $0 < \lambda < 1$ such that the following property holds: Given any $\epsilon > 0$, there exists $\delta > 0$ such that for all sequences $\{d(k)\}$ and $\{e(k)\}$ satisfying

$$|d(k)| \leq \delta \quad \text{and} \quad |e(k)| \leq \delta \quad \text{for all } k \in \mathbb{I}_{\geq 0},$$

and all $x \in C$, we have that

$$x_m(k) = x(k) + e(k) \in \mathcal{X}_N, \quad x(k) \in \mathcal{X}_N, \quad \text{for all } k \in \mathbb{I}_{\geq 0}, \quad (9a)$$

$$|\phi_{ed}(k; x)| \leq b\lambda^k |x| + \epsilon, \quad \text{for all } k \in \mathbb{I}_{\geq 0}. \quad (9b)$$
Feasibility issue

- The warm start $\tilde{u}$ is feasible for the *predicted* successor state

\[ \tilde{x}^+ = f(x_m, u(0; x_m)) \]

i.e., $(\tilde{x}^+, \tilde{u}) \in \mathbb{Z}_N$, 

- But it may not be feasible for the *measured* successor state

\[ x_m^+ = f(x, u(0; x_m)) + d + e^+ \]

(The *true* successor state, which is unknown, is 
\[ x^+ = f(x, u(0; x_m)) + d. \]

- So we have to resolve this infeasibility online. We make the following assumption

**Assumption 10**

For any $x, x' \in \mathcal{X}_N$ and $u \in \mathcal{U}_N(x)$, there exists $u' \in \mathcal{U}_N(x')$ such that
\[ |u - u'| \leq \sigma(|x - x'|) \] for some $\mathcal{K}$-function $\sigma(\cdot)$. 

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Among various options for finding $p$, we consider the following feasibility problem (notice that $\tilde{x}^+$ is known):

Find $p$ s.t. $\tilde{u} + p \in \mathcal{U}_N(x_m^+)$ and $|p| \leq \sigma(|x_m^+ - \tilde{x}^+|)$,

\begin{equation}
(10)
\end{equation}

**Proposition 11**

*Under Assumption 10, for any $(\tilde{x}^+, \tilde{u}) \in \mathbb{Z}_N$ and $x_m^+ \in \mathcal{X}_N$, the set of solutions to (10) is nonempty.*
Main results

Theorem 12 (SRES of suboptimal MPC)

Under Assumptions 1, 2.1, 3, 4, and 10, the origin of the perturbed closed-loop system

\[ x^+ \in F_{ed}(x) \]

is SRES on \( C_\rho \).
Main results

Figure: Sketch of the main sets involved in SRES.
To this aim, we replace Assumption 2.1 with 2.2.
Assumption 10 will not be necessary, whereas Assumptions 1, 3 and 4 (with $V_N(\cdot)$ replaced by $V^\beta_N(\cdot)$ later defined) are required.
We modify the cost function as follows:

$$V^\beta_N(x, u) = \sum_{k=0}^{N-1} \ell(\phi(k; x, u), u(k)) + \beta V_f(\phi(N; x, u))$$
Modified suboptimal MPC algorithm in absence of state constraints

\[ u^+ \in \mathbb{U}^N \quad \text{(11a)} \]

\[ V_N^\beta(x^+, u^+) \leq V_N^\beta(x^+, \tilde{u}) \quad \text{(11b)} \]

\[ V_N^\beta(x^+, u^+) \leq \beta V_f(x^+) \quad \text{when } x^+ \in r\mathbb{B} \quad \text{(11c)} \]

\[ \bar{Z}_r = \{(x, u) \in \mathbb{R}^n \times \mathbb{U}^N \mid V_N^\beta(x, u) \leq \bar{V}, \quad \text{and } V_N^\beta(x, u) \leq \beta V_f(x) \text{ if } x \in r\mathbb{B}\} \]

\[ \bar{X}_0 = \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{U}^N \text{ such that } (x, u) \in \bar{Z}_r\} \quad \text{(12)} \]
Nominal exponential stability

We have the following nominal stability result.

**Theorem 13 (ES without state constraints)**

*Under Assumptions 1, 2.2, 3, and 4, the origin of the closed-loop system*

\[ x^+ \in \{ f(x, u) \mid u \in \kappa_N(x) \} \]

*is ES on \( \mathbb{X}_0 \).*
Robust stability in absence of state constraints

\[ G_{ed}(z) = \{ u^+ | u^+ \in U^N, V_N^B(x^+_m, u^+) \leq V_N^B(x^+_m, \tilde{u}), \]
\[ V_N^B(x^+_m, u^+) \leq \beta V_f(x^+_m) \text{ if } x^+_m \in rB \} \]

\[ \bar{V}_N^\rho(z_m) = \max_{e \in \rho B} V_N^\beta(z) \text{ s.t. } z = z_m - (e, 0) \quad (13a) \]
\[ \tilde{S}_\rho = \{ z_m \in \tilde{Z}_r | \bar{V}_N^\rho(z_m) \leq \bar{V} \} \quad (13b) \]
\[ \bar{C}_\rho = \{ x \in \mathbb{R}^n | x = x_m - e, e \in \rho B, \exists u \text{ s.t. } (x_m, u) \in \tilde{S}_\rho \} \quad (14) \]
Robust stability in absence of state constraints

Theorem 14 (SRES without state constraints)

*Under Assumptions 1, 2.2, 3 and 4, the origin of the closed-loop system*

\[ x^+ \in F_{\text{ed}}(x) \]

*is SRES on \( \bar{C}_\rho \).*
The ideal, but unachievable, result would be that the robust region of attraction under nonzero disturbances is the entire nominal feasible set, $\mathcal{X}_N$.

This ideal result is approached reasonably closely, however, when excluding state constraints!

When $|d|, |e| \to 0$, $\tilde{C}_\rho \to \mathcal{X}_0$ and SRES holds over a set approaching the admissible set of initial conditions.

These results for robustness of suboptimal nonlinear MPC all apply to cooperative, distributed MPC.
Now let’s return to the setting of distributed, nonlinear MPC, and deal with the nonconvexity issue.

Two criteria in design:

1. the optimizers should not rely on a central coordinator
2. the exchange of information between the subsystems and the iteration of the subsystem optimizations should be able to terminate before convergence without compromising closed-loop properties.
Consider the optimization

\[
\min_u V(u) \quad \text{s.t.} \quad u \in \mathbb{U}
\]

We require approximate solutions to the following suboptimizations at iterate \( p \geq 0 \) for all \( i \in \mathbb{I}_{1:M} \)

\[
\overline{u}_i^p = \arg \min_{u_i \in \mathbb{U}_i} V(u_i, u_{-i}^p)
\]

in which \( u_{-i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_M) \).

Define the step \( v_i^p = \overline{u}_i^p - u_i^p \).
To choose the stepsize $\alpha_i^p$, each suboptimizer initializes the stepsize with $\bar{\alpha}_i$:

$$V(u^p) - V(u_i^p + \alpha_i^p v_i^p, u_{-i}^p) \geq -\sigma \alpha_i^p \nabla_i V(u^p)' v_i^p$$

in which $\sigma \in (0, 1)$.

After all suboptimizers finish the backtracking process, they exchange steps. Each suboptimizer forms a candidate step

$$u_i^{p+1} = u_i^p + w_i \alpha_i^p v_i^p \quad \forall i \in \mathbb{I}_{1:M}$$

\footnote{Armijo rule: (Bertsekas, 1999, p.230)}
Check the following inequality, which tests if $V(u^P)$ is convex-like

$$V(u^{p+1}) \leq \sum_{i \in \mathbb{I}_1:M} w_i V(u_i^P + \alpha_i^P v_i^P, u_{-i}^P)$$  \hspace{1cm} (15)$$

in which $\sum_{i \in \mathbb{I}_1:M} w_i = 1$ and $w_i > 0$ for all $i \in \mathbb{I}_1:M$.

If the condition above is not satisfied, then we find the direction with the worst cost improvement

$$i_{\text{max}} = \arg \max_i \{ V(u_i^P + \alpha_i^P v_i^P, u_{-i}^P) \}$$

and eliminate this direction by setting $w_{i_{\text{max}}}$ to zero and repartitioning the remaining $w_i$ so that they sum to 1.

At worst, condition (15) is satisfied with one direction only.
Lemma 15 (Feasibility)

Given a feasible initial condition, the iterates $u^p$ are feasible for all $p \geq 0$.

Lemma 16 (Objective decrease)

The objective function decreases at every iterate, that is, $V(u^{p+1}) \leq V(u^p)$.

Lemma 17 (Convergence)

Every accumulation point of the sequence $\{u^p\}$ is stationary.
Figure: Nonconvex function optimized with Distributed nonconvex optimization algorithm
Distributed nonlinear cooperative control

- We have the following local models

\[ x_1^+ = f_1(x_1, x_2, u_1, u_2) \quad x_2^+ = f_2(x_1, x_2, u_1, u_2) \]  \hspace{1cm} (16)

- Collect these models to form the plantwide model

\[ x^+ = f(x_1, x_2, u_1, u_2) = f(x, u) \]

in which

\[
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
=\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
\]

\[
f(x, u) = \begin{bmatrix}
  f_1(x_1, x_2, u_1, u_2) \\
  f_2(x_1, x_2, u_1, u_2)
\end{bmatrix}
\]

for which \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, \) and \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n. \)

- At each time step \( k, \) we require the inputs to satisfy

\[ u_1(k) \in U_1 \quad u_2(k) \in U_2 \quad k \in I_{0:N-1} \]

in which each \( U_i \in \mathbb{R}^{m_i} \) is compact, convex, and contains the origin in its interior.
The objective function for each subsystem $i \in \mathbb{I}_{1:2}$ is defined

$$V_i(x(0), u_1, u_2) = \sum_{k=0}^{N-1} \ell_i(x_i(k), u_i(k)) + V_{if}(x(N))$$

We define plantwide objective

$$V(x_1(0), x_2(0), u_1, u_2) = \rho_1 V_1(x(0), u_1, u_2) + \rho_2 V_2(x(0), u_1, u_2)$$

To simplify notation we use $V(x, u)$ for the plantwide objective.

**Assumption 18**

For each $i \in \mathbb{I}_{1:2}$, there exists a $K_\infty$ function $\alpha_i(\cdot)$ such that

$$\ell_i(x_i, u_i) \geq \alpha_i(|x_i|) \quad \forall (x_i, u_i) \in \mathbb{R}^{n_i} \times \mathbb{U}_i$$

(17)
Denote the plantwide terminal penalty $V_f(x) = \rho_1 V_{1f}(x) + \rho_2 V_{2f}(x)$.

We define the terminal region $\mathbb{X}_f$ to be a sublevel set of $V_f$. For $a > 0$, define

$$\mathbb{X}_f = \{x \mid V_f(x) \leq a\}$$

**Assumption 19**

The plantwide terminal penalty $V_f(\cdot)$ satisfies

$$\alpha_f(|x|) \leq V_f(x) \leq \gamma_f(|x|) \quad \forall x \in \mathbb{X}_f$$

in which $\alpha_f(\cdot)$ and $\gamma_f(\cdot)$ are $K_\infty$ functions.
Distributed nonlinear cooperative control

- Defining \( \ell(x, u) = \rho_1 \ell_1(x_1, u_1) + \rho_2 \ell_2(x_2, u_2) \), we require the following stability assumption.

**Assumption 20**

The terminal cost \( V_f(\cdot) \) satisfies

\[
\min_{(u_1, u_2) \in U_1 \times U_2} \left\{ V_f(f(x, u_1, u_2)) + \ell(x, u) \right\} \leq V_f(x) \quad \forall x \in X_f
\]

- This assumption implies that there exists a \( \kappa_{if}(x) \in U_i \) for all \( i \in \mathbb{I}_{1:2} \) such that

\[
V_f(f(x, \kappa_{1f}(x), \kappa_{2f}(x))) + \ell(x, \kappa_{1f}(x), \kappa_{2f}(x)) \leq V_f(x) \quad (18)
\]

s.t. \( f(x, \kappa_{1f}(x), \kappa_{2f}(x)) \in X_f \)
Removing the terminal constraint

- To avoid coupled input constraints, must remove the terminal state constraint.
- For some $\beta \geq 1$, define the modified objective function

\[
V^\beta(x, u) = \sum_{k=0}^{N-1} \ell(x(k), u(k)) + \beta V_f(x(N))
\]  

(19)

- Define the set of admissible initial $(x, u)$ pairs as

\[
Z_0 = \{(x, u) \in X \times U^N \mid V^\beta(x, u) \leq \bar{V}, \phi(N; x, u) \in X_f\}
\]  

(20)

- The set of initial states $X_0$ is the projection of $Z_0$ onto $X$

\[
X_0 = \{x \in X \mid \exists u \text{ such that } (x, u) \in Z_0\}
\]
Proposition 21 (Terminal constraint satisfaction)

Let \( \{(x(k), u(k)) \mid k \in I_{\geq 0}\} \) denote the set of states and control sequences generated by the suboptimal system. There exists a \( \overline{\beta} > 1 \) such that for all \( \beta \geq \overline{\beta} \), if \( (x(0), u(0)) \in \mathbb{Z}_0 \), then \( (x(k), u(k)) \in \mathbb{Z}_0 \) with \( \phi(N; x(k), u(k)) \in X_f \) for all \( k \in I_{\geq 0} \).

For the remainder of the talk, we replace the plantwide objective with the modified objective \( V(\cdot) \leftarrow V^{\overline{\beta}}(\cdot) \) and hence the terminal constraint is satisfied.
Cooperative control algorithm

- Let $\tilde{u} \in \mathbb{U}$ be the initial condition for the cooperative MPC algorithm such that $\phi(N; x(0), \tilde{u}) \in \mathbb{X}_f$.
- At each iterate $p$, an approximate solution of the following optimization problem is found

$$\begin{align*}
\min_{\mathbf{u}} & \quad V(x_1(0), x_2(0), u_1, u_2) \\
\text{s.t.} & \quad x_1^+ = f_1(x_1, x_2, u_1, u_2) \\
& \quad x_2^+ = f_2(x_1, x_2, u_1, u_2) \\
& \quad u_i \in \mathbb{U}_i^N \quad \forall i \in \mathbb{I}_{1:2} \\
& \quad |u_i| \leq \delta_i(|x_i(0)|) \quad \text{if } x(0) \in \mathcal{B}r \quad \forall i \in \mathbb{I}_{1:2}
\end{align*}$$

(21a) (21b) (21c) (21d) (21e)
Cooperative control algorithm

Let the input sequence returned by distributed nonconvex optimization algorithm be \( \mathbf{u}^\mathcal{P}(x, \tilde{u}) \).

The first input of this sequence \( \kappa^\mathcal{P}(x(0)) = u^\mathcal{P}(0; x(0), \tilde{u}) \) is injected into the plant and the state is moved forward.

To reinitialize the algorithm at the next sampling time, we define the warm start

\[
\tilde{\mathbf{u}}_1^+ = \{ u_1(1), u_1(2), \ldots, u_1(N - 1), \kappa_{1f}(x(N)) \}
\]

\[
\tilde{\mathbf{u}}_2^+ = \{ u_2(1), u_2(2), \ldots, u_2(N - 1), \kappa_{2f}(x(N)) \}
\]

in which \( x(N) = \phi(N; x(0), \mathbf{u}_1, \mathbf{u}_2) \).
Nominal stability results

Theorem 22 (Asymptotic stability)

Let Assumptions 18-20 hold and let $V(\cdot) \leftarrow V\overline{\beta}(\cdot)$ by Proposition 21. Then for every $x(0) \in \mathbb{X}_N$, the origin is asymptotically stable for the closed-loop system $x^+ = f(x, \kappa \bar{p}(x))$. 
A nonlinear example

- Consider the unstable nonlinear system
  \[
  x_1^+ = x_1^2 + x_2 + u_1^3 + u_2 \\
  x_2^+ = x_1 + x_2^2 + u_1 + u_2^3
  \]

  with initial condition \((x_1, x_2) = (3, -3)\).

- For this example, we use the stage cost
  \[
  \ell_1(x_1, u_1) = \frac{1}{2} (x_1' Q_1 x_1 + u_1' R_1 u_1) \\
  \ell_2(x_2, u_2) = \frac{1}{2} (x_2' Q_2 x_2 + u_2' R_2 u_2)
  \]

- For the simulation we choose the parameters
  \[
  Q = I \quad R = I \quad N = 2 \quad \bar{p} = 3 \quad \mathbb{U}_i = [-2.5, 2.5] \quad \forall i \in \mathbb{I}_{1:2}
  \]
Distributed nonlinear cooperative control

Figure: State trajectory ($\bar{p} = 3$)

Figure: Centralized state trajectory ($\bar{p} = 10$)
Distributed nonlinear cooperative control

Figure: Input trajectory ($\bar{\rho} = 3$)

Figure: Centralized input trajectory ($\bar{\rho} = 10$)
Distributed nonlinear cooperative control

Figure: Open-loop cost to go versus time on the closed-loop trajectory for different numbers of iterations.
Distributed nonlinear cooperative control

Figure: Contours of $V$ with $N = 1$ for $k = 0$ with $(x_1(0), x_2(0)) = (3, -3)$. Iterations of the subsystem controllers with initial condition $(u_1^0, u_2^0) = (0, 0)$. 
Figure: Terminal region. $\mathbb{X}_t$ are the points in which the terminal controller is stabilizing and $\mathbb{X}_f = \{x \mid V_f(x) \leq 0.485\} \subseteq \mathbb{X}_t$ is the terminal region.
Further Reading I


Acknowledgments

The author is indebted to Luo Ji of the University of Wisconsin and Ganzhou Wang of the Universität Stuttgart for their help in organizing the material and preparing the overheads.
Questions or Comments?