NAME: ______________________________

Instructions: The exam is open book and open notes. Write your name on the exam. Work both problems. You have two hours. Hand in your exam as well as your solution at the end.

Exercise 1. Transient heat conduction in a slab. 50 points

Consider the transient heat conduction equation

$$\rho \hat{C}_P \frac{\partial T}{\partial t} = -\nabla \cdot (-k \nabla T)$$

We have a one-dimensional slab with ends located at $x = \pm L$. The slab is initially at uniform temperature T_0. Just after $t = 0$, the two ends are immediately raised to temperature T_1 and held at this temperature. We wish to find the transient solution $T(x,t)$ for this problem.

(a) Write the PDE and (three) boundary conditions for this situation, i.e. conditions at $x = L$, $x = -L$, and $t = 0$. How many parameters appear in this problem?

(b) Choose nondimensional temperature, spatial position, and time variables as follows

$$\Theta = \frac{T - T_0}{T_1 - T_0} \qquad z = \frac{x}{L} \qquad \tau = \frac{tk}{\rho \hat{C}_P L^2}$$

Express the PDE and BCs in these nondimensional variables. How many parameters appear in this problem?

(c) Take the Laplace transform of the PDE, apply the boundary conditions and show

$$\overline{\Theta}(z,s) = \frac{\cosh(\sqrt{s}z)}{s \cosh \sqrt{s}}$$

(d) For what s values in the complex plane is $\overline{\Theta}(z,s)$ singular?

(e) Invert the transform and find $\Theta(z,\tau)$

(f) Show that $\Theta(z,\tau)$ satisfies the PDE and boundary conditions at $z = \pm 1$. Does the solution satisfy the initial condition? How would you check this?

(g) What is the steady state, $\Theta_s(z)$, i.e. take the limit of $\Theta(z,\tau)$ as $\tau \to \infty$.

Exercise 2. Heat conduction in a cylinder and a sphere. 50 pts.

Let's change the body to a cylinder and a sphere and see what happens. Again assume the body is initially at uniform temperature T_0. Just after $t = 0$, the outer boundary at $r = R$ is immediately raised to temperature T_1 and held at this temperature. We wish to find the transient solution $T(r, t)$ for these problems.

(a) Write the PDE and (three) boundary conditions for the **cylindrical** body, i.e. conditions at $r = R$, $r = 0$, and $t = 0$. How many parameters appear in this problem?

(b) Choose nondimensional temperature, radial position, and time variables as follows

$$\Theta = \frac{T - T_0}{T_1 - T_0} \qquad \xi = \frac{r}{R} \qquad \tau = \frac{tk}{\rho \hat{C}_P R^2}$$

Express the PDE and BCs in these nondimensional variables. How many parameters appear in this problem?

(c) Take the Laplace transform of the PDE, apply the boundary conditions and find $\overline{\Theta}(\xi, s)$ for the cylinder. You do not need to invert this transform.

(d) Write the PDE and (three) boundary conditions for the **spherical** body, i.e. conditions at $r = R$, $r = 0$, and $t = 0$. How many parameters appear in this problem?

(e) Choose the same nondimensional temperature, radial position, and time variables as follows

$$\Theta = \frac{T - T_0}{T_1 - T_0} \qquad \xi = \frac{r}{R} \qquad \tau = \frac{tk}{\rho \hat{C}_P R^2}$$

Express the PDE and BCs in these nondimensional variables. How many parameters appear in this problem?

(f) Take the Laplace transform of the PDE, apply the boundary conditions and find $\overline{\Theta}(\xi, s)$ for the sphere. You do not need to invert this transform.