NAME: ____________________________

Instructions: The exam is closed book and closed notes. Write your name on the exam. Work **any two of the three** problems. You have 50 minutes. Hand in your exam as well as your solution at the end.

Exercise 1. Norm and matrix rotation. 50 points

Given the following A matrix

$$A = \begin{bmatrix} 0.46287 & 0.11526 \\ 0.53244 & 0.34359 \end{bmatrix}$$

invoking `[u,s,v]=svd(A)` in MATLAB or Octave produces

```
u =
  -0.59540  -0.80343
  -0.80343   0.59540

s =
   0.78328   0.00000
   0.00000   0.12469

v =
  -0.89798  -0.44004
  -0.44004   0.89798
```

(a) What vector x of unit norm *maximizes* $\|Ax\|$? How large is $\|Ax\|$ for this x?

(b) What vector x of unit norm *minimizes* $\|Ax\|$? How large is $\|Ax\|$ for this x?

(c) What is the definition of $\|A\|$? What is the value of $\|A\|$ for this A?

(d) Denote the columns of `v` by v_1 and v_2. Draw a sketch of the unit circle traced by x as it travels from $x = v_1$ to $x = v_2$ and the corresponding curve traced by Ax.

(e) Let's find an A, if one exists, that rotates all $x \in \mathbb{R}^2$ counterclockwise by θ radians. What do you choose for the singular values σ_1 and σ_2? Choose $v_1 = e_1$ and $v_2 = e_2$ for the V matrix in which e_i, $i = 1, 2$ is the ith unit vector What do you want u_1 and u_2 to be for this rotation by θ radians. Form the product USV' and determine the A matrix that performs this rotation.

Exercise 2. Network of four isomerization reactions. 50 pts.

Consider the set of reversible, first-order reactions

$$A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} B \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} C \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} D \underset{k_{-4}}{\overset{k_4}{\rightleftharpoons}} E$$

taking place in a well-mixed, batch reactor. The reactions are all elementary reactions with correpsonding first-order rate expressions. Let the concentration of the species be stacked in a column vector

$$c = \begin{bmatrix} c_A \\ c_B \\ c_C \\ c_D \\ c_E \end{bmatrix}$$

(a) Write the mass balance for the well-mixed, batch reactor of constant volume

$$\frac{dc}{dt} = Kc$$

What is K for this problem?

(b) What is the solution of this mass balance for initial condition $c(0) = c_0$. What calculation do you do to find out if this solution is stable?

(c) Determine the rank of matrix K (hint: focus on the *rows* of K). Justify your answer. From the fundamental theorem of linear algebra, what is the dimension of the null space of K?

(d) What is the condition for a steady-state solution of the model? Is the steady state unique? Why or why not?

Exercise 3. Use the source, Luke. 50 pts.

Someone in your research group wrote a computer program that takes an n-vector input, $x \in \mathbb{R}^n$ and returns an m-vector output, $y \in \mathbb{R}^m$.

$$y = f(x)$$

All we know is that the function f is *linear*.

The code was compiled and now the source code has been lost; the author has graduated and won't respond to our email. We need to create the source code for function f so we can compile it for our newly purchased hardware, which no longer runs the old compiled code. To help us accomplish this task, all we can do is execute the function on the old hardware.

(a) How many function calls do you need to make before you can write the source code for this function?

(b) What inputs do you choose, and how do you construct the linear function f from the resulting outputs?

(c) To make matters worse, your advisor has a hot new project idea that requires you to write a program to evaluate the inverse of this linear function,

$$x = f^{-1}(y)$$

He has asked you if this is possible. How do you respond? Give a complete answer about the existence and uniqueness of x given y.