Postface to “Model Predictive Control: Theory and Design”

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The goal of this postface is to point out and comment upon recent MPC papers and issues pertaining to topics covered in the first printing of the monograph by Rawlings and Mayne (2009). We have tried to group the recent MPC literature by the relevant chapter in that reference. This compilation is selective and not intended to be a comprehensive summary of the current MPC research literature, but we welcome hearing about other papers that the reader feels should be included here.[1]

Chapter 1. Getting Started with Model Predictive Control

Offset-free control. In Section 1.5.2, Disturbances and Zero Offset, conditions are given that ensure zero offset in chosen control variables in the presence of plant/model mismatch under *any choices* of stabilizing regulator and stable estimator. In particular, choosing the number of integrating disturbances equal to the number of measurements, $n_d = p$, achieves zero offset independently of estimator and regulator tuning. A recent contribution by Maeder, Borrelli, and Morari (2009) tackles the issue of achieving offset free performance when choosing $n_d < p$. As pointed out by Pannocchia and Rawlings (2003), however, choosing $n_d < p$ also means that the gain of the estimator depends on the regulator tuning. Therefore, to maintain offset free performance, the estimator tuning must be changed if the regulator tuning is changed. Maeder et al. (2009) give design procedures for choosing estimator and regulator parameters simultaneously to achieve zero offset in this situation.

Chapter 2. Model Predictive Control — Regulation

MPC stability results with the KL definition of asymptotic stability. Since Lyapunov’s foundational work, asymptotic stability traditionally has been defined with two fundamental conditions: (i) local stability and (ii) attractivity. Control and systems texts using this classical definition include Khalil (2002, p. 112) and Vidyasagar (1993, p. 141). The classical definition was used mainly in stating and proving the stability theorems appearing in the Appendix B corresponding to the first printing of the text. Recently, however, a stronger definition of asymptotic stability, which we refer to here as the “KL” definition, has started to become popular. These two definitions are compared and contrasted in a later section of this postface (see Appendix B – Stability Theory). We used the KL definition of asymptotic stability to

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define state estimator stability in Chapter 4 (see Definition 4.6, for example). We outline here how to extend the main MPC stability results of Chapter 2 to apply under this stronger definition of asymptotic stability.[2]

In many MPC applications using nonlinear models, it is straightforward to obtain an upper bound on the MPC value function on a small set $\mathbb{X}_f$ containing the origin in its interior. For example, this bound can be established when the linearization of the system is stabilizable at the origin. But it may be difficult to extend this upper bound to cover the entire stabilizable set $\mathcal{X}_N$. But we require this upper bound to apply standard Lyapunov stability theory to the MPC controller. Therefore, we next wish to extend the upper bounding $\mathcal{K}_\infty$ function $\alpha_2(\cdot)$ from the local set $\mathbb{X}_f$ to all of $\mathcal{X}_N$, including the case when $\mathcal{X}_N$ is unbounded. Given the local upper bounding $\mathcal{K}_\infty$ function on $\mathbb{X}_f$, the necessary and sufficient condition for function $V(\cdot)$ to have an upper bounding $\mathcal{K}_\infty$ function on all of $\mathcal{X}_N$ is that $V(\cdot)$ is *locally* bounded on $\mathcal{X}_N$, i.e., $V(\cdot)$ is bounded on every compact subset of $\mathcal{X}_N$. See Appendix B of this note for a statement and proof of this result. So we first establish that $V_N^0(\cdot)$ is locally bounded on $\mathcal{X}_N$.

Proposition 1 (MPC value function is locally bounded)**.** *Suppose Assumptions 2.2 and 2.3 hold. Then $V_N^0(\cdot)$ is locally bounded on $\mathcal{X}_N$.*

Proof. Let X be an arbitrary compact subset of $\mathcal{X}_N$. The function $V_N : \mathbb{R}^n \times \mathbb{R}^{Nm} \to \mathbb{R}_{\geq 0}$ is defined and continuous and therefore has an upper bound on the compact set $X \times \mathbb{U}^N$. Since $\mathcal{U}_N(x) \subset \mathbb{U}^N$ for all $x \in \mathcal{X}_N$, $V_N^0 : \mathcal{X}_N \to \mathbb{R}_{\geq 0}$ has the same upper bound on X. Since X is arbitrary, we have established that $V_N^0(\cdot)$ is locally bounded on $\mathcal{X}_N$. ■

We next extend Proposition 2.18 by removing the assumption that $\mathcal{X}_N$ is compact.

Proposition 2 (Extension of upper bound to $\mathcal{X}_N$)**.** *Suppose that Assumptions 2.2, 2.3, 2.12, and 2.13 hold and that $\mathbb{X}_f$ contains the origin in its interior. If there exists a $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that $V_N^0(x) \leq \alpha(|x|)$ for all $x \in \mathbb{X}_f$, then there exists another $\mathcal{K}_\infty$ function $\beta(\cdot)$ such that $V_N^0(x) \leq \beta(|x|)$ for all $x \in \mathcal{X}_N$.*

Proof. From the definition of $\mathcal{X}_N$ and Assumptions 2.12 and 2.13, we have that $\mathbb{X}_f \subseteq \mathcal{X}_N$. From Proposition 2.11, we have that the set $\mathcal{X}_N$ is closed, and this proposition therefore follows directly from Proposition 11 in Appendix B of this note. ■

Remark 3. The extension of Proposition 2.18 to unbounded $\mathcal{X}_N$ also removes the need to assume $\mathcal{X}_N$ is bounded in Proposition 2.19.

Finally, we can establish Theorem 2.22 under the stronger "KL" definition of asymptotic stability.

Theorem 4 (Asymptotic stability with unbounded region of attraction)**.** *Suppose $\mathcal{X}_N \subset \mathbb{R}^n$ and $\mathbb{X}_f \subset \mathcal{X}_N$ are positive invariant for the system $x^+ = f(x)$, that $\mathbb{X}_f \subset \mathcal{X}_N$ is closed and contains the origin in its interior, and that there exist a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and two $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that*

$$V(x) \geq \alpha_1(|x|) \qquad \forall x \in \mathcal{X}_N \tag{1}$$

$$V(x) \leq \alpha_2(|x|) \qquad \forall x \in \mathbb{X}_f \tag{2}$$

$$V(f(x)) - V(x) \leq -\alpha_1(|x|) \quad \forall x \in \mathcal{X}_N \tag{3}$$

[2]The authors would like to thank Andy Teel of UCSB for helpful discussion of these issues.

Text	Postface	Summary of change
Proposition 2.18	Proposition 2	Removes boundedness of $\mathcal{X}_N$
Proposition 2.19	Remark 3	Removes boundedness of $\mathcal{X}_N$
Theorem 2.22	Theorem 4	Asymptotic stability with stronger KL definition
Definition B.6	Definition 9	Classical to KL definition of asymptotic stability
Theorem B.11	Theorem 12	Lyapunov function and KL definition
Definition B.9 (e)	Definition 13	Asymptotic stability with KL definition (constrained)
Theorem B.13	Theorem 14	Lyapunov function and KL definition (constrained)

Table 1: Extensions of MPC stability results in Chapter 2 and Appendix B.

Then the origin is asymptotically stable under Definition 9 with a region of attraction $\mathcal{X}_N$ for the system $x^+ = f(x)$.

Proof. Proposition 2 extends the local upper bound in (2) to all of $\mathcal{X}_N$ and Theorem 14 then gives asymptotic stability under Definition 13. Both Theorem 14 and Definition 13 appear in Appendix B of this note. ∎

A summary of these extensions to the results of Chapter 2 and Appendix B is provided in Table 1.

Positive invariance under control law $\boldsymbol{\kappa_N(\cdot)}$. Proposition 2.11 correctly states that the set $\mathcal{X}_N$ is positive invariant for the closed-loop system $x^+ = f(x, \kappa_N(x))$. The proof follows from (2.11), and is stated in the text as:

> That $\mathcal{X}_N$ is positive invariant for $x^+ = f(x, \kappa_N(x))$ follows from (2.11), which shows that $\kappa_N(\cdot)$ steers every $x \in \mathcal{X}_N$ into $\mathcal{X}_{N-1} \subseteq \mathcal{X}_N$.

But notice that this same argument establishes that $\mathcal{X}_{N-1}$ is also positive invariant for the closed-loop system, a fact that does not seem to have been noticed previously. Since $\mathcal{X}_{N-1} \subseteq \mathcal{X}_N$, this statement is a tighter characterization of the positive invariance property. This tighter characterization is sometimes useful when establishing robust stability for systems with discontinuous $V_N^0(\cdot)$, such as Example 2.8. Among the feasibility sets, $\mathcal{X}_j, j = 0, 1, \ldots, N$, the set $\mathcal{X}_N$ is the largest positive invariant set and $\mathcal{X}_{N-1}$ is the smallest positive invariant set for $x^+ = f(x, \kappa_N(x))$; none of the other feasibility sets, $\mathcal{X}_j, j = 0, 1, \ldots, N-2$, are necessarily positive invariant for $x^+ = f(x, \kappa_N(x))$ for all systems satisfying the given assumptions. A modified Proposition 2.11 reads as follows.

Proposition 2.11' (Existence of solutions to DP recursion). *Suppose Assumptions 2.2 and 2.3 hold. Then*

(a) For all $j \in \mathbb{I}_{\geq 0}$, the cost function $V_j(\cdot)$ is continuous in $\mathcal{Z}_j$, and, for each $x \in \mathcal{X}_j$, the control constraint set $\mathcal{U}_j(x)$ is compact and a solution $\mathbf{u}^0(x) \in \mathcal{U}_j(x)$ to $\mathbb{P}_j(x)$ exists.

(b) If $\mathcal{X}_0 := \mathbb{X}_f$ is control invariant for $x^+ = f(x, u)$, $u \in \mathbb{U}$, then, for each $j \in \mathbb{I}_{\geq 0}$, the set $\mathcal{X}_j$ is also control invariant, $\mathcal{X}_j \supseteq \mathcal{X}_{j-1}$, $0 \in \mathcal{X}_j$, and $\mathcal{X}_j$ is closed.

(c) In addition, the sets $\mathcal{X}_j$ and $\mathcal{X}_{j-1}$ are positive invariant for $x^+ = f(x, \kappa_j(x))$ for all $j \in \mathbb{I}_{\geq 1}$.

Unreachable setpoints, strong duality, and dissipativity. Unreachable setpoints are discussed in Section 2.9.3. It is known that the optimal MPC value function in this case is not decreasing and is therefore not a Lyapunov function for the closed-loop system. A recent paper by Diehl, Amrit, and Rawlings (2011) has shown that a modified MPC cost function, termed rotated cost, *is* a Lyapunov function for the unreachable setpoint case and other more general cost functions required for optimizing process economics. A strong duality condition is shown to be a sufficient condition for asymptotic stability of economic MPC with nonlinear models.

This result is further generalized in the recent paper Angeli, Amrit, and Rawlings (2012). Here a dissipation inequality is shown to be sufficient for asymptotic stability of economic MPC with nonlinear models. This paper also shows that MPC is better than optimal periodic control for systems that are not optimally operated at steady state.

Unbounded input constraint sets. Assumption 2.3 includes the restriction that the input constraint set $\mathbb{U}$ is compact (bounded and closed). This basic assumption is used to ensure existence of the solution to the optimal control problem throughout Chapter 2. If one is interested in an MPC theory that handles an unbounded input constraint set $\mathbb{U}$, then one can proceed as follows. First modify Assumption 2.3 by removing the boundedness assumption on $\mathbb{U}$.

Assumption 5 (Properties of constraint sets – unbounded case)**.** The sets $\mathbb{X}$, $\mathbb{X}_f$, and $\mathbb{U}$ are closed, $\mathbb{X}_f \subseteq \mathbb{X}$; each set contains the origin.

Then, to ensure existence of the solution to the optimal control problem, consider the cost assumption on page 154 in the section on nonpositive definite stage costs, slightly restated here.

Assumption 6 (Stage cost lower bound)**.** Consider the following two lower bounds for the stage cost.

(a)

$$\begin{aligned} \ell(y,u) &\geq \alpha_1(\left|(y,u)\right|) && \text{for all } y \in \mathbb{R}^p, u \in \mathbb{R}^m \\ V_f(x) &\leq \alpha_2(|x|) && \text{for all } x \in \mathbb{X}_f \end{aligned}$$

in which $\alpha_1(\cdot)$ is a $\mathcal{K}_\infty$ function.

(b)

$$\begin{aligned} \ell(y,u) &\geq c_1 \left|(y,u)\right|^a && \text{for all } y \in \mathbb{R}^p, u \in \mathbb{R}^m \\ V_f(x) &\leq c_2 \left|x\right|^a && \text{for all } x \in \mathbb{X}_f \end{aligned}$$

in which $c_1, c_2, a > 0$.

Finally, assume that the system is input/output-to-state stable (IOSS). This property is given in Definition 2.40 (or Definition B.42). We can then state an MPC stability theorem that applies to the case of unbounded constraint sets.

Theorem 7 (MPC stability – unbounded constraint sets)**.**

(a) Suppose that Assumptions 2.2, 5, 2.12, 2.13, and 6(a) hold and that the system $x^+ = f(x,u), y = h(x)$ is IOSS. Then the origin is asymptotically stable (under Definition 9) with a region of attraction $\mathcal{X}_N$ for the system $x^+ = f(x, \kappa_N(x))$.

(b) Suppose that Assumptions 2.2, 5, 2.12, 2.13, and 6(b) hold and that the system $x^+ = f(x,u), y = h(x)$ is IOSS. Then the origin is exponentially stable with a region of attraction $\mathcal{X}_N$ for the system $x^+ = f(x, \kappa_N(x))$.

In particular, setting up the MPC theory with these assumptions subsumes the LQR problem as a special case.

Example 1: The case of the linear quadratic regulator

Consider the linear, time invariant model $x^+ = Ax + Bu, y = Cx$ with quadratic penalties $\ell(y,u) = (1/2)(y'Qy + u'Ru)$ for both the finite and infinite horizon cases. What do the assumptions of Theorem 7(b) imply in this case? Compare these assumptions to the standard LQR assumptions listed in Exercise 1.20 (b).

Assumption 2.2 is satisfied for $f(x,u) = Ax + Bu$ for all $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$; we have $\mathbb{X} = \mathbb{R}^n$, and $\mathbb{U} = \mathbb{R}^m$. Assumption 6(b) implies that $Q > 0$ and $R > 0$. The system being IOSS implies that (A,C) is detectable (see Exercise 4.5). We can choose $\mathbb{X}_f$ to be the stabilizable subspace of (A,B) and Assumption 2.13 is satisfied. The set $\mathcal{X}_N$ contains the system controllability information. The set $\mathcal{X}_N$ is the stabilizable subspace of (A,B), and we can satisfy Assumption 6(a) by choosing $V_f(x) = (1/2)x'\Pi x$ in which Π is the solution to the steady-state Riccati equation for the stabilizable modes of (A,B).

In particular, if (A,B) is stabilizable, then $\mathbb{X}_f = \mathbb{R}^n$, $\mathcal{X}_N = \mathbb{R}^n$ for all $N \in \mathbb{I}_{0:\infty}$, and V_f can be chosen to be $V_f(x) = (1/2)x'\Pi x$ in which Π is the solution to the steady-state Riccati equation (1.19). The horizon N can be finite or infinite with this choice of $V_f(\cdot)$ and the control law is invariant with respect to the horizon length, $\kappa_N(x) = Kx$ in which K is the steady-state linear quadratic regulator gain given in (1.19). Theorem 7(b) then gives that the origin of the closed-loop system $x^+ = f(x,\kappa_N(x)) = (A+BK)x$ is globally, exponentially stable.

The standard assumptions for the LQR with stage cost $l(y,u) = (1/2)(y'Qy + u'Ru)$ are

$$Q > 0 \quad R > 0 \quad (A,C) \text{ detectable} \quad (A,B) \text{ stabilizable}$$

and we see that this case is subsumed by Theorem 7(b). □

Chapter 6. Distributed Model Predictive Control

The recent paper (Stewart, Venkat, Rawlings, Wright, and Pannocchia, 2010) provides a compact treatment of many of the issues and results discussed in Chapter 6. Also, for plants with sparsely coupled input constraints, it provides an extension that achieves centralized optimality on convergence of the controllers' iterations.

Suboptimal MPC and inherent robustness. The recent paper (Pannocchia, Rawlings, and Wright, 2011) takes the suboptimal MPC formulation in Section 6.1.2, also discussed in Section 2.8, and establishes its *inherent robustness* to bounded process and measurement disturbances. See also the paper by Lazar and Heemels (2009), which first addressed inherent robustness of suboptimal MPC to process disturbances by (i) specifying a degree of suboptimality and (ii) using the time-varying state constraint tightening approach of Limón Marruedo, Álamo, and Camacho (2002) to achieve recursive feasibility under disturbances.

The key assumption in (Pannocchia et al., 2011) is the following.

Assumption 8. For any $x, x' \in \mathcal{X}_N$ and $\mathbf{u} \in \mathcal{U}_N(x)$, there exists $\mathbf{u}' \in \mathcal{U}_N(x')$ such that $|\mathbf{u} - \mathbf{u}'| \leq \sigma(|x - x'|)$ for some $\mathcal{K}$-function $\sigma(\cdot)$.

This assumption also implies that $V_N^0(\cdot)$ is continuous by applying Theorem C.28 in Rawlings and Mayne (2009). If state constraints are softened and the terminal stability constrained is removed by choosing a suitably increased terminal penalty, then this assumption is automatically satisfied. The conclusion of (Pannocchia et al., 2011) is that *suboptimal MPC has the same inherent robustness properties as optimal MPC.*

Nonlinear distributed MPC. A recent paper (Stewart, Wright, and Rawlings, 2011) proposes a method for handling the nonconvex optimization resulting from nonlinear plant models. The basic difficulty is that taking the convex step of the local controllers' optimizations may not decrease the plantwide cost. To overcome this problem, the following procedure is proposed.

> After all suboptimizers finish an iteration, they exchange steps. Each suboptimizer forms a candidate step
>
> $$u_i^{p+1} = u_i^p + w_i \alpha_i^p v_i^p \quad \forall i \in \mathbb{I}_{1:M} \tag{4}$$
>
> and checks the following inequality, which tests if $V(u^p)$ is convex-like
>
> $$V(u^{p+1}) \leq \sum_{i \in \mathbb{I}_{1:M}} w_i V(u_i^p + \alpha_i^p v_i^p, u_{-i}^p) \tag{5}$$
>
> in which $\sum_{i \in \mathbb{I}_{1:M}} w_i = 1$ and $w_i > 0$ for all $i \in \mathbb{I}_{1:M}$. If condition (5) is not satisfied, then we find the direction with the worst cost improvement $i_{\max} = \arg\max_i\{V(u_i^p + \alpha_i^p v_i^p, u_{-i}^p)\}$, and eliminate this direction by setting $w_{i_{\max}}$ to zero and repartitioning the remaining w_i so that they sum to 1. We then reform the candidate step (4) and check condition (5) again. We repeat until (5) is satisfied. At worst, condition (5) is satisfied with only one direction.

Notice that the test of inequality (5) does not require a coordinator. Each subsystem has a copy of the plantwide model and can evaluate the objection function independently. Therefore, the set of comparisons can be run on each controller. This computation represents a small overhead compared to a coordinating optimization.

Appendix B. Stability Theory

Asymptotic stability. For several of the stability theorems appearing in the first printing's Appendix B,[3] we used the classical definition of global asymptotic stability (GAS), given in Definition B.6. The following stronger definition of GAS has recently started to become popular.

Definition 9 (Global asymptotic stability (KL version)). The (closed, positive invariant) set $\mathcal{A}$ is *globally asymptotically stable* (GAS) for $x^+ = f(x)$ if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ such that, for each $x \in \mathbb{R}^n$

$$|\phi(i; x)|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, i) \qquad \forall i \in \mathbb{I}_{\geq 0} \tag{B.1}$$

[3]See the website www.che.wisc.edu/~jbraw/mpc for the Appendices A–C corresponding to the first printing of the text.

Notice that this inequality appears as (B.1) in Appendix B.

Teel and Zaccarian (2006) provide further discussion of these definitional issues. It is also interesting to note that although the KL definitions may have become popular only recently, Hahn (1967, p. 8) used K and L comparison functions as early as 1967 to define asymptotic stability.[4] For *continuous* $f(\cdot)$, we show in Proposition B.8 that these two definitions are equivalent. But we should bear in mind that for nonlinear models, the function $f(\cdot)$ defining the closed-loop system evolution under MPC, $x^+ = f(x, \kappa_N(x))$, may be *discontinuous* because the control law $\kappa_N(\cdot)$ may be discontinuous (see Example 2.8 in Chapter 2 for an example). Also, when using *suboptimal* MPC, the control law is a point to set map and is not a continuous function (Rawlings and Mayne, 2009, pp. 156, 417). For *discontinuous* $f(\cdot)$, the two definitions are *not* equivalent. Consider the following example to make this clear.

Example 2: Difference between asymptotic stability definitions (Teel)

Consider the *discontinuous* nonlinear scalar example $x^+ = f(x)$ with

$$f(x) = \begin{cases} \frac{1}{2}x & |x| \in [0,1] \\ \dfrac{2x}{2-|x|} & |x| \in (1,2) \\ 0 & |x| \in [2,\infty) \end{cases}$$

The origin is attractive for all $x(0) \in \mathbb{R}$, which can be demonstrated as follows. For $|x(0)| \in [0,1]$, $|x(k)| \leq (1/2)^k |x(0)|$. For $|x(0)| \in (1,2)$, $|x(1)| \geq 2$ which implies that $|x(2)| = 0$; and for $|x(0)| \in [2,\infty)$, $|x(1)| = 0$. The origin is Lyapunov stable, because if $\delta \leq 1$, then $|x(0)| \leq \delta$ implies $|x(k)| \leq \delta$ for all k. Therefore, the origin is asymptotically stable according to the classical definition.

But there is no $\mathcal{KL}$ function $\beta(\cdot)$ such that the system meets the bound for all $x(0) \in \mathbb{R}$

$$|x(k)| \leq \beta(|x(0)|, k) \qquad \forall k \in \mathbb{I}_{\geq 0}$$

Indeed, for initial conditions $|x(0)|$ slightly less than 2, the trajectory $x(k)$ becomes arbitrarily large (at $k = 1$) before converging to the origin. Therefore, the origin is *not* asymptotically stable according to the KL definition. □

Remark 10. Note that because of Proposition B.8, the function $f(\cdot)$ must be chosen to be discontinuous in this example to demonstrate this difference.

Proposition 11 (Extending local upper bounding function)**.** *Suppose the function $V(\cdot)$ is defined on X, a closed subset of $\mathbb{R}^n$, and that $V(x) \leq \alpha(|x|_{\mathcal{A}})$ for all $x \in \mathbb{X}_f$ where $\mathbb{X}_f \subseteq X$ and contains the set $\mathcal{A}$ in its interior. A necessary and sufficient condition for the existence of a $\mathcal{K}_\infty$ function $\beta(\cdot)$ satisfying $V(x) \leq \beta(|x|_{\mathcal{A}})$ for all $x \in X$ is that $V(\cdot)$ is locally bounded on X, i.e., $V(\cdot)$ is bounded on every compact subset of X.*

Proof.

[4]The authors would like to thank Mircea Lazar of Eindhoven University for helpful discussion of this history.

Sufficiency. We assume that $V(\cdot)$ is locally bounded and construct the function $\beta(\cdot)$. Because $\mathcal{A}$ lies in the interior of $\mathbb{X}_f$, there exists an $a > 0$ such that $|x|_{\mathcal{A}} \leq a$ implies $x \in \mathbb{X}_f$. For each $i \in \mathbb{I}_{\geq 1}$, let $S_i = \{x \mid |x|_{\mathcal{A}} \leq ia\}$. We define a sequence of numbers (α_i) as follows

$$\alpha_i := \sup_{S_i \cap X} V(x) + \alpha(a) + i$$

Since S_i is compact for each i and X is closed, their intersection is a compact subset of X and the values α_i exist for all $i \in \mathbb{I}_{\geq 1}$ because $V(\cdot)$ is bounded on every compact subset of X. The sequence (α_i) is strictly increasing. For each $i \in \mathbb{I}_{\geq 1}$, let the interpolating function $\phi_i(\cdot)$ be defined by

$$\phi_i(s) := (s - ia)/a \qquad s \in [ia, (i+1)a]$$

Note that $\phi_i(ia) = 0$, $\phi_i((i+1)a) = 1$, and that $\phi(\cdot)$ is affine in $[ia, (i+1)a]$. We can now define the function $\beta(\cdot)$ as follows

$$\beta(s) := \begin{cases} (\alpha_2/\alpha(a))\alpha(s) & s \in [0, a] \\ \alpha_{i+1} + \phi_i(s)(\alpha_{i+2} - \alpha_{i+1}) & s \in [ia, (i+1)a] \quad \text{for all } i \in \mathbb{I}_{\geq 1} \end{cases}$$

It can be seen that $\beta(0) = 0$, $\beta(s) \geq \alpha(s)$ for $s \in [0, a]$, that $\beta(\cdot)$ is continuous, strictly increasing, and unbounded, and that $V(x) \leq \beta(|x|_{\mathcal{A}})$ for all $x \in X$. Hence we have established the existence of a $\mathcal{K}_\infty$ function $\beta(\cdot)$ such that $V(x) \leq \beta(|x|_{\mathcal{A}})$ for all $x \in X$.

Necessity. If we assume that $V(\cdot)$ is *not* locally bounded, i.e., not bounded on some compact set $C \subseteq X$, it follows immediately that there is no (continuous and, hence, locally bounded) $\mathcal{K}_\infty$ function $\beta(\cdot)$ such that such that $V(x) \leq \beta(x)$ for all $x \in C$. ∎

Note, however, that most of the Lyapunov function theorems appearing in Appendix B also hold under the stronger KL definition of GAS. As an example, we provide a modified proof required for establishing Theorem B.11.

Theorem 12 (Lyapunov function and GAS). *Suppose $V(\cdot)$ is a Lyapunov function for $x^+ = f(x)$ and set $\mathcal{A}$ with $\alpha_3(\cdot)$ a $\mathcal{K}_\infty$ function. Then $\mathcal{A}$ is globally asymptotically stable under Definition 9.*

Proof. From (B.4) of Definition B.10, we have that

$$V(\phi(i+1;x)) \leq V(\phi(i;x)) - \alpha_3(\left|\phi(i;x)\right|_{\mathcal{A}}) \qquad \forall x \in \mathbb{R}^n \quad i \in \mathbb{I}_{\geq 0}$$

Using (B.3) we have that

$$\alpha_3(|x|_{\mathcal{A}}) \geq \alpha_3 \circ \alpha_2^{-1}(V(x)) \qquad \forall x \in \mathbb{R}^n$$

Combining these we have that

$$V(\phi(i+1;x)) \leq \sigma_1(V(\phi(i;x))) \qquad \forall x \in \mathbb{R}^n \quad i \in \mathbb{I}_{\geq 0}$$

in which

$$\sigma_1(\cdot) := (\cdot) - \alpha_3 \circ \alpha_2^{-1}(\cdot)$$

We have that $\sigma_1(\cdot)$ is continuous on $\mathbb{R}_{\geq 0}$, $\sigma_1(0) = 0$, and $0 < \sigma_1(s) < s$ for $s > 0$. But $\sigma_1(\cdot)$ may not be increasing. We modify σ_1 to achieve this property in two steps. First define

$$\sigma_2(s) := \max_{s' \in [0,s]} \sigma_1(s') \qquad s \in \mathbb{R}_{\geq 0}$$

in which the maximum exists for each $s \in \mathbb{R}_{\geq 0}$ because $\sigma_1(\cdot)$ is continuous. By its definition, $\sigma_2(\cdot)$ is nondecreasing, and we next show that $\sigma_2(\cdot)$ is continuous on $\mathbb{R}_{\geq 0}$. Assume that $\sigma_2(\cdot)$ is discontinuous at a point $c \in \mathbb{R}_{\geq 0}$. Because it is a nondecreasing function, there is a positive jump in the function $\sigma_2(\cdot)$ at c (Bartle and Sherbert, 2000, p. 150). Define [5]

$$a_1 := \lim_{s \nearrow c} \sigma_2(s) \qquad a_2 := \lim_{s \searrow c} \sigma_2(s)$$

We have that $\sigma_1(c) \leq a_1 < a_2$ or we violate the limit of σ_2 from below. Since $\sigma_1(c) < a_2$, $\sigma_1(s)$ must achieve value a_2 for some $s < c$ or we violate the limit from above. But $\sigma_1(s) = a_2$ for $s < c$ also violates the limit from below, and we have a contradiction and $\sigma_2(\cdot)$ is continuous. Finally, define

$$\sigma(s) := (1/2)(s + \sigma_2(s)) \qquad s \in \mathbb{R}_{\geq 0}$$

and we have that $\sigma(\cdot)$ is a continuous, strictly increasing, and unbounded function satisfying $\sigma(0) = 0$. Therefore, $\sigma(\cdot) \in \mathcal{K}_\infty$, $\sigma_1(s) < \sigma(s) < s$ for $s > 0$ and therefore

$$V(\phi(i+1; x)) \leq \sigma(V(\phi(i; x))) \qquad \forall x \in \mathbb{R}^n \quad i \in \mathbb{I}_{\geq 0} \tag{6}$$

Repeated use of (6) and then (B.3) gives

$$V(\phi(i; x)) \leq \sigma^i(\alpha_2(|x|_{\mathcal{A}})) \qquad \forall x \in \mathbb{R}^n \quad i \in \mathbb{I}_{\geq 0}$$

in which σ^i represents the composition of σ with itself i times. Using (B.2) we have that

$$\left|\phi(i; x)\right|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, i) \qquad \forall x \in \mathbb{R}^n \quad i \in \mathbb{I}_{\geq 0}$$

in which

$$\beta(s, i) := \alpha_1^{-1}(\sigma^i(\alpha_2(s))) \qquad \forall s \in \mathbb{R}_{\geq 0} \quad i \in \mathbb{I}_{\geq 0}$$

For all $s \geq 0$, the sequence $w_i := \sigma^i(\alpha_2(s))$ is nonincreasing with i, bounded below (by zero), and therefore converges to a, say, as $i \to \infty$. Therefore, both $w_i \to a$ and $\sigma(w_i) \to a$ as $i \to \infty$. Since $\sigma(\cdot)$ is continuous we also have that $\sigma(w_i) \to \sigma(a)$ so $\sigma(a) = a$, which implies that $a = 0$, and we have shown that for all $s \geq 0$, $\alpha_1^{-1}(\sigma^i(\alpha_2(s))) \to 0$ as $i \to \infty$. Since $\alpha_1^{-1}(\cdot)$ also is a $\mathcal{K}$ function, we also have that for all $s \geq 0$, $\alpha_1^{-1}(\sigma^i(\alpha_2(s)))$ is nonincreasing with i. We have from the properties of $\mathcal{K}$ functions that for all $i \geq 0$, $\alpha_1^{-1}(\sigma^i(\alpha_2(s)))$ is a $\mathcal{K}$ function, and can therefore conclude that $\beta(\cdot)$ is a $\mathcal{KL}$ function and the proof is complete. ■

Constrained case. Definition B.9 lists the various forms of stability for the constrained case in which we consider $X \subset \mathbb{R}^n$ to be positive invariant for $x^+ = f(x)$. In the classical definition, set $\mathcal{A}$ is asymptotically stable with region of attraction X if it is locally stable in X and attractive in X. The KL version of asymptotic stability for the constrained case is the following.

[5] The limits from above and below exist because $\sigma_2(\cdot)$ is nondecreasing (Bartle and Sherbert, 2000, p. 149). If the point $c = 0$, replace the limit from below by $\sigma_2(0)$.

Definition 13 (Asymptotic stability (constrained – KL version)). Suppose $X \subset \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$, that $\mathcal{A}$ is closed and positive invariant for $x^+ = f(x)$, and that $\mathcal{A}$ lies in the interior of X. The set $\mathcal{A}$ is *asymptotically stable with a region of attraction* X for $x^+ = f(x)$ if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ such that, for each $x \in X$

$$\left|\phi(i;x)\right|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, i) \qquad \forall i \in \mathbb{I}_{\geq 0} \tag{7}$$

Notice that we simply replace $\mathbb{R}^n$ with the set X in Definition 9 to obtain Definition 13. We then have the following result, analogous to Theorem B.13, connecting a Lyapunov function to the KL version of asymptotic stability for the constrained case.

Theorem 14 (Lyapunov function for asymptotic stability (constrained case – KL version)). *Suppose $X \subset \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$, that $\mathcal{A}$ is closed and positive invariant for $x^+ = f(x)$, and that $\mathcal{A}$ lies in the interior of X. If there exists a Lyapunov function in X for the system $x^+ = f(x)$ and set $\mathcal{A}$ with $\alpha_3(\cdot)$ a $\mathcal{K}_\infty$ function, then $\mathcal{A}$ is asymptotically stable for $x^+ = f(x)$ with a region of attraction X under Definition 13.*

The proof of this result is similar to that of Theorem 12 with $\mathbb{R}^n$ replaced by X.

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