Outline

1. Setpoint Tracking
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   - Steady-state target problem
   - Dynamic regulation problem
   - State estimation, disturbance model and zero offset

2. Further Topics in MPC
   - Distributed MPC
   - MPC with Discrete Actuators
   - Economic MPC

3. Conclusions
In this section we show how to use the MPC regulator and MHE estimator to handle different kinds of control problems, including setpoint tracking and rejecting nonzero disturbances.

It is a standard objective in applications to use a feedback controller to move the measured outputs of a system to a specified and constant setpoint. This problem is known as setpoint tracking.

In nonlinear MPC theory we can consider the case in which the system is nonlinear and constrained, but here we consider linear model MPC in which $y_{sp}$ is an arbitrary constant.
Deviation variables

- In the regulation problem we assumed that the goal was to take the state of the system to the origin. Such a regulator can be used to treat the setpoint tracking problem with a coordinate transformation.
- Denote the desired output setpoint as $y_{sp}$. Denote a steady state of the system model as $(x_s, u_s)$. The steady state satisfies

$$\begin{bmatrix} I - A & -B \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = 0$$

- For *unconstrained* systems, we also impose the requirement that the steady state satisfies $Cx_s = y_{sp}$ for the tracking problem, giving the set of equations

$$\begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_{sp} \end{bmatrix}$$

(1)
Deviations variables

- If this set of equations has a solution, we can then define deviation variables

\[ \tilde{x}(k) = x(k) - x_s \]
\[ \tilde{u}(k) = u(k) - u_s \]

- They satisfy the dynamic model

\[ \tilde{x}(k + 1) = \tilde{x}(k + 1) - x_s \]
\[ = A x(k) + B u(k) - (A x_s + B u_s) \]
\[ \tilde{x}(k + 1) = A \tilde{x}(k) + B \tilde{u}(k) \]

- The deviation variables satisfy the same model equation as the original variables! This feature holds only for linear models.
Deviation variables

The zero regulation problem applied to the system in deviation variables finds $\tilde{u}(k)$ that takes $\tilde{x}(k)$ to zero, or, equivalently, which takes $x(k)$ to $x_s$, so that at steady state, $Cx(k) = Cx_s = y_{sp}$, which is the goal of the setpoint tracking problem.

After solving the regulation problem in deviation variables, the input applied to the system is

$$u(k) = \tilde{u}(k) + u_s$$

We next discuss when we can solve (1). We also note that for constrained systems, we must impose the constraints on the steady state $(x_s, u_s)$. 

More outputs than inputs: controlled variables

- The matrix in (1) is a \((n + p) \times (n + m)\) matrix. For (1) to have a solution for all \(y_{sp}\), it is sufficient that the rows of the matrix are linearly independent.

- That requires \(p \leq m\): we require at least as many inputs as outputs with setpoints. But it is not uncommon in applications to have many more measured outputs than manipulated inputs.

- To handle these more general situations, we choose a matrix \(H\) and denote a new variable \(r = Hy\) as a selection of linear combinations of the measured outputs. The variable \(r \in \mathbb{R}^{nc}\) is known as the controlled variable.

- For cases in which \(p > m\), we choose some set of outputs \(n_c \leq m\), as controlled variables, and assign setpoints to \(r\), denoted \(r_{sp}\).
We also wish to treat systems with more inputs than outputs, $m > p$. For these cases, the solution to (1) may exist for some choice of $H$ and $r_{sp}$, but cannot be unique.

If we wish to obtain a unique steady state, then we also must provide desired values for the steady inputs, $u_{sp}$.

To handle constrained systems, we simply impose the constraints on $(x_s, u_s)$. 

Our candidate optimization problem is therefore

\[
\min_{x_s, u_s} \frac{1}{2} \left( |u_s - u_{sp}|_{R_s}^2 + |Cx_s - y_{sp}|_{Q_s}^2 \right)
\] (2a)

subject to:

\[
\begin{bmatrix}
I - A & -B \\
HC & 0
\end{bmatrix}
\begin{bmatrix}
x_s \\
u_s
\end{bmatrix} =
\begin{bmatrix}
0 \\
r_{sp}
\end{bmatrix}
\] (2b)

\[Eu_s \leq e\] (2c)

\[FCx_s \leq f\] (2d)
We make the following assumptions:

**Assumption 1 (Target feasibility and uniqueness)**

1. The target problem is feasible for the controlled variable setpoints of interest $r_{sp}$.
2. The steady-state input penalty $R_s$ is positive definite.

- Assumption 1.1 ensures that the solution $(x_s, u_s)$ exists
- Assumption 1.2 ensures that the solution is unique.
- If one chooses $n_c = 0$, then no controlled variables are required to be at setpoint, and the problem is feasible for any $(u_{sp}, y_{sp})$ because $(x_s, u_s) = (0, 0)$ is a feasible point.
Some exercises on the target problem

- Exercises 1.56 and 1.57 explore the connection between feasibility of the equality constraints and the number of controlled variables relative to the number of inputs and outputs.
- One restriction is that the number of controlled variables chosen to be offset free must be less than or equal to the number of manipulated variables and the number of measurements, $n_c \leq m$ and $n_c \leq p$. 
Dynamic regulation problem

Given the steady-state solution, we define the following multistage objective function

$$V(\tilde{x}(0), \tilde{u}) = \frac{1}{2} \sum_{k=0}^{N-1} |\tilde{x}(k)|_Q^2 + |\tilde{u}(k)|_R^2$$

s.t. $\tilde{x}^+ = A\tilde{x} + B\tilde{u}$

The initial state is

$$\tilde{x}(0) = \hat{x}(k) - x_s$$

i.e., the initial condition for the regulation problem comes from the state estimate shifted by the steady-state $x_s$. 
The regulator solves the following dynamic, zero-state regulation problem

$$\min_{\tilde{u}} V(\tilde{x}(0), \tilde{u})$$

subject to

$$E\tilde{u} \leq e - Eu_s$$
$$FC\tilde{x} \leq f - FCx_s$$

in which the constraints also are shifted by the steady state \((x_s, u_s)\).
The optimal cost and solution are $V^0(\tilde{x}(0))$ and $\tilde{u}^0(\tilde{x}(0))$.

The moving horizon control law uses the first move of this optimal sequence, $\tilde{u}^0(\tilde{x}(0)) = \tilde{u}^0(0; \tilde{x}(0))$, so the controller output is

$$u(k) = \tilde{u}^0(\tilde{x}(0)) + u_s$$

The control law is more complex than the PID control law, but the control is a function of the estimated state, and the estimated state depends on the measurements. That’s the feedback in MPC!

Designing the state estimator is crucial to good closed-loop control performance.
The assembly so far

\[ \ddot{x} + A\dot{x} + B\ddot{u} = (Q, R) \]

Block diagram:
- Regulator
- Plant
- Estimator
- Target selector

Inputs:
- \( x_s \)
- \( u_s \)
- \( \dot{x} \)

Output:
- \( y \)

Feedforward:
- \( y_{sp}, u_{sp}, r_{sp} \)
Disturbances and zero offset

- Another common objective in applications is to use a feedback controller to compensate for an unmeasured disturbance to the system with the input so the disturbance’s effect on the controlled variable is mitigated.

- This problem is known as disturbance rejection. We may wish to design a feedback controller that compensates for nonzero disturbances such that the selected controlled variables asymptotically approach their setpoints without offset.

- This property is known as zero offset. In this section we show a simple method for constructing an MPC controller to achieve zero offset.
Disturbances and zero offset

We will ensure that *if the system is stabilized in the presence of the disturbance*, then there is zero offset.

This more limited objective is similar to what one achieves when using the integral mode in proportional-integral-derivative (PID) control of an unconstrained system: either there is zero steady offset, or the system trajectory is unbounded.

In a constrained system, the statement is amended to: either there is zero steady offset, or the system trajectory is unbounded, or the system constraints are active at steady state.

In both constrained and unconstrained systems, the zero-offset property *precludes* one undesirable possibility: the system settles at an unconstrained steady state, and the steady state displays offset in the controlled variables.
So why does the PI controller have zero offset?

- Here’s the control law

\[ u(t) = k_c \left( e(t) + \frac{1}{\tau I} \int_0^t e(t') dt' \right), \quad e = y_{sp} - y \]

- If the tracking error goes to a (nonzero) constant, \( e(t) \to e_s \), then \( u(t) \to \infty \) as \( t \to \infty \) because of the integral term.

- If we turn off the integrator

\[ u_s = k_c e_s \]

and we expect offset with a proportional controller.

- In PI, we obtain zero offset by integrating the tracking error. In MPC we will *not* integrate the tracking error. But we will integrate instead the *model error*.

- We can show that is also sufficient to remove offset. And we won’t have windup when inputs saturate.
A simple method to compensate for an unmeasured disturbance is to
1. model the disturbance
2. use the measurements and model to estimate the disturbance
3. find the inputs that minimize the effect of the disturbance on the controlled variables.

The choice of disturbance model is motivated by the zero-offset goal. To achieve offset-free performance we augment the system state with an *integrating* disturbance $d$ driven by the process noise $w$

$$d(k+1) = d(k) + w(k)$$  \hspace{1cm} (3)$$

$d$ integrates the driving noise $w$

$$d(k) = w(0) + w(1) + \cdots + w(k-1)$$
This choice is motivated by the works of Davison and Smith (1971, 1974); Qiu and Davison (1993) and the Internal Model Principle of Francis and Wonham (1976).

To remove offset, one designs a control system that can remove asymptotically constant, nonzero disturbances (Davison and Smith, 1971), (Kwakernaak and Sivan, 1972, p.278).

To accomplish this end, the original system is augmented with a replicate of the constant, nonzero disturbance model, (3). Thus the states of the original system are moved to cancel the effect of the disturbance on the controlled variables.
The augmented system model used for the state estimator is given by

\[
\begin{bmatrix}
    x \\
    d
\end{bmatrix}^+ =
\begin{bmatrix}
    A & B_d \\
    0 & I
\end{bmatrix}
\begin{bmatrix}
    x \\
    d
\end{bmatrix} +
\begin{bmatrix}
    B \\
    0
\end{bmatrix} u + w
\] (4a)

\[
y =
\begin{bmatrix}
    C & C_d
\end{bmatrix}
\begin{bmatrix}
    x \\
    d
\end{bmatrix} + \nu
\] (4b)

We are free to choose how the integrating disturbance affects the states and measured outputs through the choice of \( B_d \) and \( C_d \).

The only restriction is that the augmented system is detectable. That restriction can be easily checked using the following result.
Restrictions on the disturbance model

Lemma 2 (Detectability of the augmented system)

The augmented system (4) is detectable if and only if the nonaugmented system $(A, C)$ is detectable, and the following condition holds:

\[
\text{rank} \begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} = n + n_d
\]

Corollary 3 (Dimension of the disturbance)

The maximal dimension of the disturbance $d$ in (4) such that the augmented system is detectable is equal to the number of measurements, that is

\[n_d \leq p\]

A pair of matrices $(B_d, C_d)$ such that (5) is satisfied always exists.
The state and the additional integrating disturbance are estimated from the plant measurement using a Kalman filter designed for the augmented system.

The variances of the stochastic disturbances $w$ and $v$ may be treated as adjustable parameters or found from input-output measurements (Odelson, Rajamani, and Rawlings, 2006).
Overview of the final assembly

\[
\ddot{x}^+ = A\ddot{x} + B\ddot{u}
\]

\[(Q, R)\]

\[
\begin{bmatrix}
\hat{x} \\
\hat{d}
\end{bmatrix}^+ =
\begin{bmatrix}
A & B_d \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{d}
\end{bmatrix}
+
\begin{bmatrix}
B \\
0
\end{bmatrix} u +
\begin{bmatrix}
L_x \\
L_d
\end{bmatrix}
\left(y - \begin{bmatrix} C & C_d \end{bmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{d}
\end{bmatrix}\right)
\]
The estimator provides $\hat{x}(k)$ and $\hat{d}(k)$ at each time $k$. The best forecast of the steady-state disturbance using (3) is simply

$$\hat{d}_s = \hat{d}(k)$$
The nonzero disturbance affects the steady-state target

The steady-state target problem is therefore modified to account for the nonzero disturbance $\hat{d}_s$

$$\min \frac{1}{2} \left( |u_s - u_{sp}|^2_{R_s} + |Cx_s + C_d \hat{d}_s - y_{sp}|^2_{Q_s} \right)$$  \hspace{1cm} (6a)$$

subject to:

$$\begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} B_d \hat{d}_s \\ r_{sp} - HC_d \hat{d}_s \end{bmatrix}$$  \hspace{1cm} (6b)$$

$$Eu_s \leq e$$  \hspace{1cm} (6c)$$

$$FCx_s \leq f - FC_d \hat{d}_s$$  \hspace{1cm} (6d)$$
Effect of the disturbance

Comparing (2) to (6), we see the disturbance model affects the steady-state target determination in four places.

1. The output target is modified in (6a) to account for the effect of the disturbance on the measured output \((y_{sp} \rightarrow y_{sp} - C_d \hat{d}_s)\).

2. The output constraint in (6d) is similarly modified \((f \rightarrow f - FC_d \hat{d}_s)\).

3. The system steady-state relation in (6b) is modified to account for the effect of the disturbance on the state evolution \((0 \rightarrow B_d \hat{d}_s)\).

4. The controlled variable target in (6b) is modified to account for the effect of the disturbance on the controlled variable \((r_{sp} \rightarrow r_{sp} - HC_d \hat{d}_s)\).
No change to the regulation problem, only the target!

- Given the steady-state target, the same dynamic regulation problem as presented in the tracking section is used for the regulator.
- In other words, the regulator is based on the deterministic system \((A, B)\) in which the current state is \(\hat{x}(k) - x_s\) and the goal is to take the system to the origin.
Lemma 4 (Offset-free control)

Consider a system controlled by the MPC algorithm as shown in the figure. The target problem (6) is assumed feasible. Augment the system model with a number of integrating disturbances equal to the number of measurements \((n_d = p)\); choose any \(B_d \in \mathbb{R}^{n \times p}\), \(C_d \in \mathbb{R}^{p \times p}\) such that

\[
\text{rank} \begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} = n + p
\]

If the plant output \(y(k)\) goes to steady state \(y_s\), the closed-loop system is stable, and constraints are not active at steady state, then there is zero offset in the controlled variables, that is

\[
Hy_s = r_{sp}
\]
Remarks on offset

- The proof of this lemma is given in Pannocchia and Rawlings (2003).
- It may seem surprising that the number of integrating disturbances must be equal to the number of *measurements* used for feedback rather than the number of *controlled variables* to guarantee offset-free control.
- To gain insight into the reason, consider the disturbance part (bottom half) of the Kalman filter equations shown in the figure

\[
\hat{d}^+ = \hat{d} + L_d \left( y - \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} \right)
\]

- Because of the integrator, the disturbance estimate cannot converge until

\[
L_d \left( y - \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} \right) = 0
\]
Let the output prediction error be

\[ L_d e = 0 \quad e = y - \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} \]

- If we choose \( n_d = n_c < p \), then the number of columns of \( L_d \) is greater than the number of rows and \( L_d e = 0 \) does not force \( e = 0 \).
- In general, we require the output prediction error to be zero to achieve zero offset independently of the regulator tuning.
- For \( L_d e = 0 \) to force \( e = 0 \), we require \( n_d \geq p \).
- Since we also know \( n_d \leq p \) from Corollary 3, we conclude \( n_d = p \).
Notice also that Lemma 4 does not require that the plant output be generated by the model. The theorem applies regardless of what generates the plant output. If the plant is identical to the system plus disturbance model assumed in the estimator, then the conclusion can be strengthened.

In the nominal case without measurement or process noise \((w = 0, v = 0)\), for a set of plant initial states, the closed-loop system converges to a steady state and the feasible steady-state target is achieved leading to zero offset in the controlled variables.

Characterizing the set of initial states in the region of convergence, and stabilizing the system when the plant and the model differ, are treated in Chapters 3 and 5 of (Rawlings and Mayne, 2009).

We conclude this section with a nonlinear example that demonstrates the use of Lemma 4.
Example 5

We consider a simple building with one uniform zone. The air in the zone is partially vented and recirculated across a cooling coil.
System Description

- The controlled variables are $T_z$ and $h_z$, the temperature and humidity of the zone. The additional state variable is $T_w$, the temperature of the walls.
- The manipulated variables are $\dot{m}$, the circulation rate of air, and $\rho$, the makeup fraction of ambient air.
- The ambient conditions, $T_o$ and $h_o$, as well as the internal heat/humidity gains $q_z$, $p_z$, and $q_w$ act as unmeasured disturbances.
- The steady-state operating point is chosen as $T_{ss}^z = 22.2$ °C and $h_{ss}^z = 0.45$.
- The controller and estimator will use a linear approximation of the full nonlinear system that is discretized with sample time $\Delta = 1$ min.
Mix: Part of return air is vented and made up from the ambient

\[ T_r = \rho T_o + (1 - \rho) T_z \]
\[ h_r = \rho h_o + (1 - \rho) h_r \]

Cool: Air is passed over a cooling coil, reducing temperature and humidity

\[ T_s = \frac{\dot{m} T_r + \lambda T_c}{\dot{m} + \lambda} \]
\[ h_s = \frac{\dot{m} h_r + \mu h_c}{\dot{m} + \mu} \]

Balance: Mass and energy balances give ODE model

\[ m_z c_z \frac{dT_z}{dt} = -\dot{m} c (T_z - T_s) - k_z (T_z - T_w) + q_z \]
\[ m_z \frac{dh_z}{dt} = -\dot{m} (h_z - h_s) + p_z \]
\[ m_w c_w \frac{dT_w}{dt} = -k_z (T_w - T_z) - k_o (T_w - T_o) + q_w \]
Substituting the expressions from the previous slide, a nonlinear model is obtained:

\[
m_z c_z \frac{dT_z}{dt} = -\dot{m}c \left( T_z - \frac{\dot{m}(\rho T_o + (1 - \rho) T_z)}{\dot{m} + \lambda} \right) - k_z(T_z - T_w) + q_z
\]

\[
m_z \frac{dh_z}{dt} = -\dot{m} \left( h_z - \frac{\dot{m}(\rho h_o + (1 - \rho) h_r) + \mu h_c}{\dot{m} + \mu} \right) + p_z
\]

\[
m_w c_w \frac{dT_w}{dt} = -k_z(T_w - T_z) - k_o(T_w - T_o) + q_w
\]

The linear model is then given by

\[
x^+ = Ax + Bu + B_p p
\]

\[
x = \begin{pmatrix} T_z - T_{z\text{ss}} \\ h_z - h_{z\text{ss}} \\ T_w - T_{w\text{ss}} \end{pmatrix}, \quad u = \begin{pmatrix} \dot{m} - \dot{m}_{ss} \\ \rho - \rho_{ss} \end{pmatrix}, \quad p = \begin{pmatrix} T_o - T_{o\text{ss}} \\ h_o - h_{o\text{ss}} \\ q_z - q_{z\text{ss}} \\ p_z - p_{z\text{ss}} \\ q_w - q_{w\text{ss}} \end{pmatrix}
\]
Output disturbance model

- A common first choice for a disturbance model is an output disturbance model, $B_d = 0, C_d = I$
- To make sure that this choice is valid, we check the rank condition of Lemma 4
  - Our system does satisfy the rank condition
  - This disturbance model will either remove offset or destabilize the system
- Note that systems with integrators (e.g., the level of a tank as an output) will fail the rank condition with an output disturbance model
  - It is impossible to separate the effect of the integrating state and the integrating disturbance
  - Thus, the system is undetectable
- We simulate the system’s closed-loop response to a step disturbance in $q_z$
Results: output disturbance model

Figure 1: Closed-loop trajectory using the output disturbance model. Both outputs return to their setpoints following the step disturbance.
Improved performance can often be achieved if the chosen disturbance model matches the nature of the disturbances actually affecting the system.

As a first attempt, we model our integrating disturbances after $q_z$ and $q_w$, the direct heat gains of the walls and zone.

\[
B_d = B_p \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_d = 0
\]

- We take the third and fifth columns of $B_p$ as $B_d$
- Using this disturbance model, we simulate the system’s response to the same step disturbance in $q_z$. 
Figure 2: Closed-loop trajectory using undetectable disturbance model. Although zone temperature quickly returns to its setpoint, there is significant offset in zone humidity.
What happened?

- Had we checked the rank condition from Lemma 4, we would have found that it is violated
  - Both $q_z$ and $q_w$ affect the system by increasing $T_z$ without affecting $h_z$
  - Thus, it is not possible to uniquely determine which disturbance is causing the output behavior
- Although the estimator can (reasonably) accurately determine $q_z$, it cannot compensate for nonlinearity in humidity
  - Thus, there is sustained offset
- Instead, we choose $q_z$ and $p_z$ as the disturbances to model

$$
B_d = B_p \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C_d = 0
$$

- This choice does satisfy Lemma 4.
Figure 3: Closed-loop trajectory using good disturbance model. System returns to setpoint faster than with output disturbance model.
Conclusions

- Although the output disturbance model does not work for all systems, it is often a good first choice.
- Accurately modeling disturbances can improve closed-loop performance.
Lab Exercises

- Exercise 1.56
- Exercise 1.57
- Exercise 1.58
- Exercise 1.60
- Reproduce Figure 4 of Rajamani, Rawlings, and Qin (2009)
- Reproduce Figure 6 of Rajamani et al. (2009)
Distributed MPC—System architecture

Overall system

$S_1$  
$\mathcal{N}_1 = \{2, 3\}$

$S_2$  
$\mathcal{N}_2 = \{1\}$

$S_3$  
$\mathcal{N}_3 = \{2\}$

$u_1$  
$y_1$

$u_2$  
$y_2$

$u_3$  
$y_3$
Three options for distributed MPC

Decentralized MPC

1. Ignore interactions, pair, and hope.

\[
\begin{align*}
\text{MPC}_1 & \quad \min_{u_1} V_1 \mid \mathcal{N}_1 = \emptyset \\
\text{MPC}_2 & \quad \min_{u_2} V_2 \mid \mathcal{N}_2 = \emptyset
\end{align*}
\]
Noncooperative MPC

Model interactions, communicate, and hope.

\[ \text{MPC}_1 \min_{u_1} V_1 \]
\[ u_1 \]
\[ \text{MPC}_2 \min_{u_2} V_2 \]
\[ u_2 \]
Cooperative MPC (Rawlings and Mayne, 2009, Chapter 6)

Model interactions, communicate, and cooperate. Provides stability and performance guarantee.

\[ \begin{align*}
\text{MPC}_1 & \quad \text{min}_{u_1} \rho_1 V_1 + \rho_2 V_2 \\
\text{MPC}_2 & \quad \text{min}_{u_2} \rho_1 V_1 + \rho_2 V_2 \\
\end{align*} \]
MPC with Discrete Actuators

In addition to continuous inputs $u$, systems also have discrete inputs $v$ that are constrained to be integers. This allows many different types of systems to be described with linear constraints.

- Piecewise-affine systems with state-dependent dynamics.
- Switched systems with input-dependent dynamics.
- Semi-continuous variables (e.g., $x \in \{0\} \cup [1, 2]$) as in the figure.
- Scheduling models with discrete decisions.
Example: Switched System

Consider the following switched-system dynamics:

\[
\frac{dx}{dt} = A(v)x + Bu, \quad B = \begin{bmatrix} 1/10 & 0 \\ 0 & 1/10 \end{bmatrix}
\]

Figure 4: System dynamics for \( u = (0, 0) \), \( v = 1 \), \( A(1) = \begin{bmatrix} -1/5 & -1 \\ 2 & -1/5 \end{bmatrix} \)

Figure 5: System dynamics for \( u = (0, 0) \), \( v = 0 \), \( A(0) = \begin{bmatrix} -1/5 & -2 \\ 1 & -1/5 \end{bmatrix} \)
We can solve the optimal control problem with these integer variables.

Figure 6: Two control trajectories of a switched system. $\Delta t = 0.2$, $u \in [-1, 1] \times [-1, 1]$, $v \in \{0, 1\}$, $N = 20$. Line colors show value of binary variable.
Economic MPC

Typically, setpoints are chosen in a different layer by some economic optimization

$$\min_{x,u} \mathcal{L}_{\text{econ}}(x, u) \quad \text{s.t.} \quad x = f(x, u)$$

- The optimization is static, i.e., only looking for steady states
- The models may be different, but it’s the same general idea
Economic MPC

Typically, setpoints are chosen in a different layer by some economic optimization

$$\min_{x, u} \ell_{econ}(x, u) \quad \text{s.t.} \quad x = f(x, u)$$

- The optimization is static, i.e., only looking for steady states
- The models may be different, but it’s the same general idea

If $\ell_{econ}(\cdot)$ is what we care about, we can optimize it *dynamically.*
Economic MPC

Typically, setpoints are chosen in a different layer by some economic optimization

\[ \min_{x,u} \ell_{\text{econ}}(x, u) \quad \text{s.t.} \quad x = f(x, u) \]

- The optimization is static, i.e., only looking for steady states
- The models may be different, but it’s the same general idea

If \( \ell_{\text{econ}}(\cdot) \) is what we care about, we can optimize it \textit{dynamically}.

Standard MPC

- In tracking MPC, objective function is symmetric and arbitrary
Economic MPC

Typically, setpoints are chosen in a different layer by some economic optimization

\[
\min_{x,u} \ l_{\text{econ}}(x, u) \quad \text{s.t.} \quad x = f(x, u)
\]

- The optimization is static, i.e., only looking for steady states
- The models may be different, but it’s the same general idea

If \( l_{\text{econ}}(\cdot) \) is what we care about, we can optimize it \textit{dynamically}.

Standard MPC

\[ \text{Standard MPC diagram} \]

Economic MPC

\[ \text{Economic MPC diagram} \]

- In tracking MPC, objective function is symmetric and arbitrary
- In economic MPC, objective function is tangible and likely asymmetric
Example: Time-varying Building Cooling

Temperature must be maintained within preset bounds.
Running the chiller uses electricity with time-varying price.
Want to maintain temperature bounds at lowest possible cost.

\[
\frac{dT}{dt} = -k(T - T_{amb}) + q_{amb} + q_{ch} - q_{tank}
\]

\[
\frac{ds}{dt} = -\sigma s + q_{tank}
\]

\[
q_{min} \leq q \leq q_{max}, \quad q_{tank} \leq q_{ch}
\]

\[
x := (T, s)
\]

\[
u := (q_{ch}, q_{tank})
\]

\[
d := (T_{amb}, q_{amb})
\]
Cost Function and Terminal Constraints

We will compare economic MPC to tracking the economically optimal periodic solution with standard tracking MPC.

**Stage Costs**

$$\ell_{\text{econ}}(x, u, t) := c(t)q(t)$$

$$\ell_{\text{track}}(x, u, t) := |x(t) - x_p(t)|^2_Q + |u(t) - u_p(t)|^2_R$$

- Horizon $N = 24$
- Periodic $c, T_{\text{amb}}, q_{\text{amb}}$
- Weights $Q = \text{diag}(1, 0.1), R = I$
- Optimal periodic solution used as terminal constraint

**Periodic Solution**

$$\min_{x,u} \sum_{k=0}^{N-1} \ell_{\text{econ}}(x, u, k)$$

s.t. $x(k+1) = Ax(k) + Bu(k) + f(k)$

$x(0) = x(N)$

- Initial condition $x(0)$ is free but must equal $x(N)$
- Denote solution $x_p, u_p$
Figure 7: Economically optimal periodic solution. Blue lines show system quantities, red lines show ambient, and dashed lines show bounds. Cyan line in “Cooling” plot shows storage charge/discharge.
Figure 8: Example closed-loop trajectory. Cost of **Periodic solution** is $113.31. For **economic MPC**, cost is $94.66. For **tracking MPC**, cost is $105.73.
Conclusion

- We have extended standard theory to handle discrete actuators for
  - Optimal (and suboptimal) robust stabilization of an equilibrium point.
  - Stabilization of a periodic solution (tracking).
  - Economic MPC with a periodic constraint.

Based on these results we offer the following conjecture:

**Theorem 6 (Folk theorem)**

Any result that holds for standard MPC holds also for MPC with discrete actuators. (Rawlings and Risbeck, 2017)

Applications include a rich class of commercial building energy optimization problems.

A current challenge is to develop better software tools for efficient, reliable online solution of the mixed-integer optimal control problems.

See casadi.org
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Further reading


Further reading II


Exercise 1.54: Where is the steady state?

Consider the two-input, two-output system

\[
\begin{align*}
A &= \begin{bmatrix}
0.5 & 0 & 0 & 0 \\
0 & 0.6 & 0 & 0 \\
0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0.6
\end{bmatrix} \\
B &= \begin{bmatrix}
0.5 & 0 \\
0 & 0.4 \\
0.25 & 0 \\
0 & 0.6
\end{bmatrix} \\
C &= \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\end{align*}
\]

(a) The output setpoint is \(y_{sp} = \begin{bmatrix} 1 & -1 \end{bmatrix}'\) and the input setpoint is \(u_{sp} = \begin{bmatrix} 0 & 0 \end{bmatrix}'\). Calculate the target triple \((x_s, u_s, y_s)\). Is the output setpoint feasible, i.e., does \(y_s = y_{sp}\)?

(b) Assume only input one \(u_1\) is available for control. Is the output setpoint feasible? What is the target in this case using \(Q_s = I\)?

(c) Assume both inputs are available for control but only the first output has a setpoint, \(y_{1t} = 1\). What is the solution to the target problem for \(R_s = I\)?

Exercise 1.55: Detectability of integrating disturbance models

(a) Prove Lemma 1.8; the augmented system is detectable if and only if the system \((A, C)\) is detectable and

\[
\text{rank} \begin{bmatrix}
I - A & -B_d \\
C & C_d
\end{bmatrix} = n + n_d
\]

(b) Prove Corollary 1.9; the augmented system is detectable only if \(n_d \leq p\).

Exercise 1.56: Unconstrained tracking problem

(a) For an unconstrained system, show that the following condition is sufficient for feasibility of the target problem for any \(r_{sp}\).

\[
\text{rank} \begin{bmatrix}
I - A \\
HC & 0
\end{bmatrix} = n + n_c
\]  \hspace{1cm} (1.69)

(b) Show that (1.69) implies that the number of controlled variables without offset is less than or equal to the number of manipulated variables and the number of measurements, \(n_c \leq m\) and \(n_c \leq p\).

(c) Show that (1.69) implies the rows of \(H\) are independent.

(d) Does (1.69) imply that the rows of \(C\) are independent? If so, prove it; if not, provide a counterexample.

(e) By choosing \(H\), how can one satisfy (1.69) if one has installed redundant sensors so several rows of \(C\) are identical?

Exercise 1.57: Unconstrained tracking problem for stabilizable systems

If we restrict attention to stabilizable systems, the sufficient condition of Exercise 1.56 becomes a necessary and sufficient condition. Prove the following lemma.
Lemma 1.14 (Stabilizable systems and feasible targets). Consider an unconstrained, stabilizable system \((A, B)\). The target is feasible for any \(r_{sp}\) if and only if
\[
\text{rank} \begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} = n + n_c
\]

Exercise 1.58: Existence and uniqueness of the unconstrained target
Assume a system having \(p\) controlled variables \(z = Hx\), with setpoints \(r_{sp}\), and \(m\) manipulated variables \(u\), with setpoints \(u_{sp}\). Consider the steady-state target problem
\[
\min_{x,u} \frac{1}{2} (u - u_{sp})' R (u - u_{sp}) \quad R > 0
\]
subject to
\[
\begin{bmatrix} I - A & -B \\ H & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ r_{sp} \end{bmatrix}
\]
Show that the steady-state solution \((x, u)\) exists for any \((r_{sp}, u_{sp})\) and is unique if
\[
\text{rank} \begin{bmatrix} I - A & -B \\ H & 0 \end{bmatrix} = n + p \quad \text{rank} \begin{bmatrix} I - A \\ H \end{bmatrix} = n
\]

Exercise 1.59: Choose a sample time
Consider the unstable continuous time system
\[
\frac{dx}{dt} = Ax + Bu \quad y = Cx
\]
in which
\[
A = \begin{bmatrix} -0.281 & 0.935 & 0.035 & 0.008 \\ 0.047 & -0.116 & 0.053 & 0.383 \\ 0.679 & 0.519 & 0.030 & 0.067 \\ 0.679 & 0.831 & 0.671 & -0.083 \end{bmatrix} \quad B = \begin{bmatrix} 0.687 \\ 0.589 \\ 0.930 \\ 0.846 \end{bmatrix} \quad C = I
\]
Consider regulator tuning parameters and constraints
\[
Q = \text{diag}(1, 2, 1, 2) \quad R = 1 \quad N = 10 \quad |x| \leq \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}
\]
(a) Compute the eigenvalues of \(A\). Choose a sample time of \(\Delta = 0.04\) and simulate the MPC regulator response given \(x(0) = \begin{bmatrix} -0.9 & -1.8 & 0.7 & 2 \end{bmatrix}'\) until \(t = 20\). Use an ODE solver to simulate the continuous time plant response. Plot all states and the input versus time.
Now add an input disturbance to the regulator so the control applied to the plant is \(u_d\) instead of \(u\) in which
\[
u_d(k) = (1 + 0.1w_1)u(k) + 0.1w_2
\]
and \(w_1\) and \(w_2\) are zero mean, normally distributed random variables with unit variance. Simulate the regulator’s performance given this disturbance. Plot all states and \(u_d(k)\) versus time.
Exercise 1.60: Disturbance models and offset
Consider the following two-input, three-output plant discussed in Example 1.11

\[ x^+ = Ax + Bu + B_p p \]
\[ y = Cx \]

in which

\[ A = \begin{bmatrix} 0.2681 & -0.00338 & -0.00728 \\ 9.703 & 0.3279 & -25.44 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.00537 & 0.1655 \\ 1.297 & 97.91 \\ 0 & -6.637 \end{bmatrix}, \quad B_p = \begin{bmatrix} -0.1175 \\ 69.74 \\ 6.637 \end{bmatrix} \]

The input disturbance \( p \) results from a reactor inlet flowrate disturbance.

(a) Since there are two inputs, choose two outputs in which to remove steady-state offset. Build an output disturbance model with two integrators. Is your augmented model detectable?

(b) Implement your controller using \( p = 0.01 \) as a step disturbance at \( k = 0 \). Do you remove offset in your chosen outputs? Do you remove offset in any outputs?

(c) Can you find any two-integrator disturbance model that removes offset in two outputs? If so, which disturbance model do you use? If not, why not?

Exercise 1.61: MPC, PID and time delay
Consider the following first-order system with time delay shown in Figure 1.13

\[ g(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad k = 1, \tau = 1, \theta = 5 \]

Consider a unit step change in setpoint \( y_{sp} \), at \( t = 0 \).