Optimal control unchained

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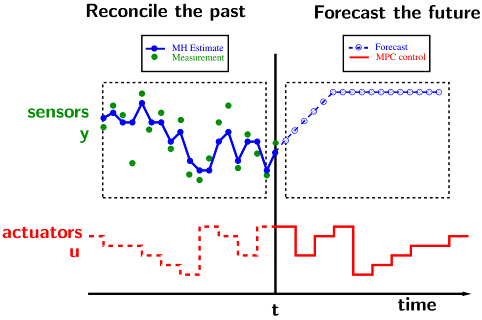
September 30, 2010
50 Years of Nonlinear Control and Optimization —
A Historical Workshop
Royal Society, London, UK

Outline

1. Optimal control, optimal feedback control, and model predictive control (MPC)
2. Industrial impact of these ideas
3. Using MPC to optimize plant economics
4. Some comments on David Mayne

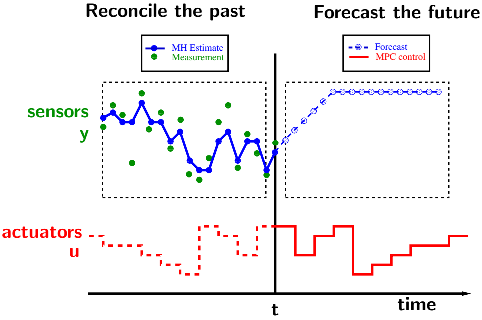
Predictive control



$$\min_{u(t)} \int_0^T |y_{sp} - g(x,u)|_Q^2 + |u_{sp} - u|_R^2 \, dt$$

$$\dot{x} = f(x,u)$$
$$x(0) = x_0 \quad \text{(given)}$$
$$y = g(x,u)$$

State estimation



$$\min_{x_0, w(t)} \int_{-T}^{0} |y - g(x,u)|_R^2 + |\dot{x} - f(x,u)|_Q^2 \, dt$$

$$\dot{x} = f(x,u) + w \quad \text{(process noise)}$$
$$y = g(x,u) + v \quad \text{(measurement noise)}$$

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- Optimal feedback control sees limited industrial application during this period.

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— Lee and Markus (1967)
Foundations of Optimal Control Theory

Our notion of very rapidly changed radically from 1960 to 1985.

Large industrial success story!

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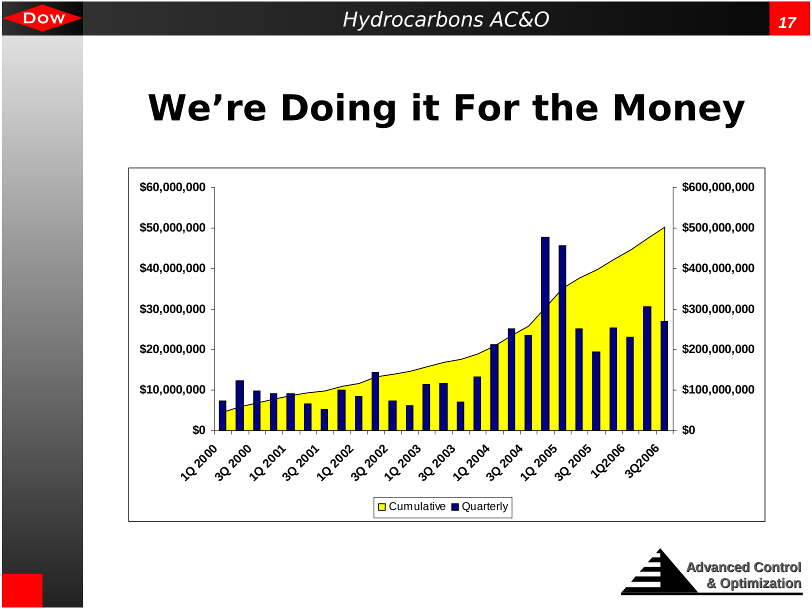
Impact for 13 ethylene plants (Starks and Arrieta, 2007)



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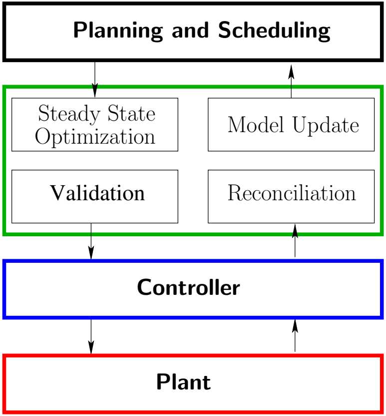
- Do we have tools to optimize dynamic *economic* operation?

Optimizing economics: Current industrial practice



- Two layer structure
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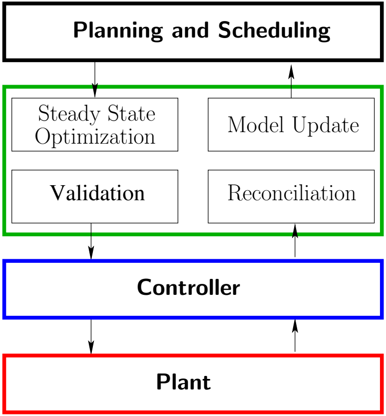
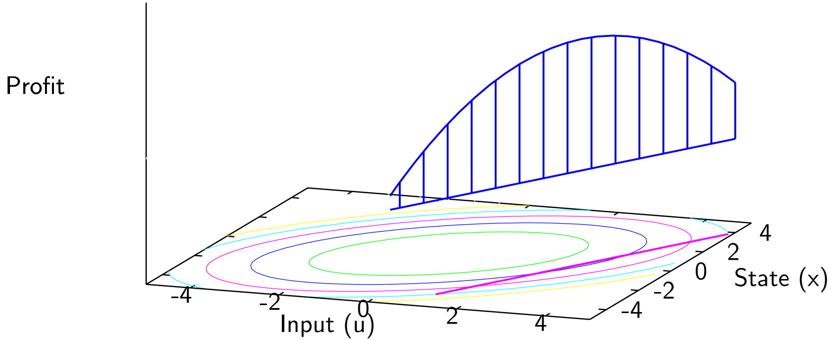


- Two layer structure
- Drawbacks
  - Inconsistent models
  - Re-identify linear model as setpoint changes
  - Time scale separation may not hold
  - Economics unavailable in dynamic layer

Optimizing economics: what's desirable?



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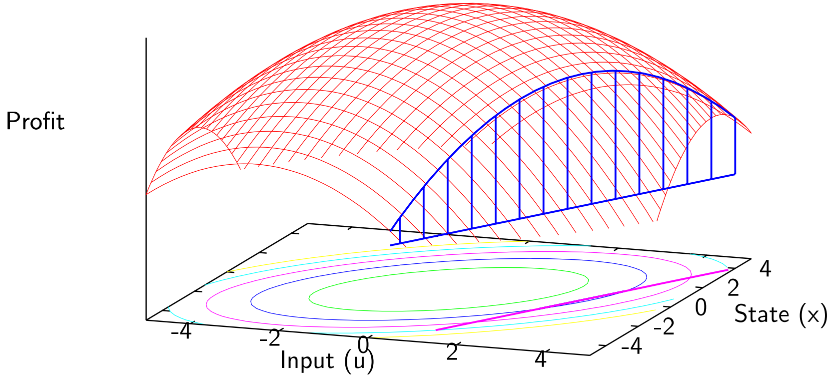

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- Solution: (x_s, u_s)

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$$\text{std–MPC:} \quad \tilde{\ell}(x, u) = |x - x_s|_Q^2 + |u - u_s|_R^2 \qquad \text{–or–}$$
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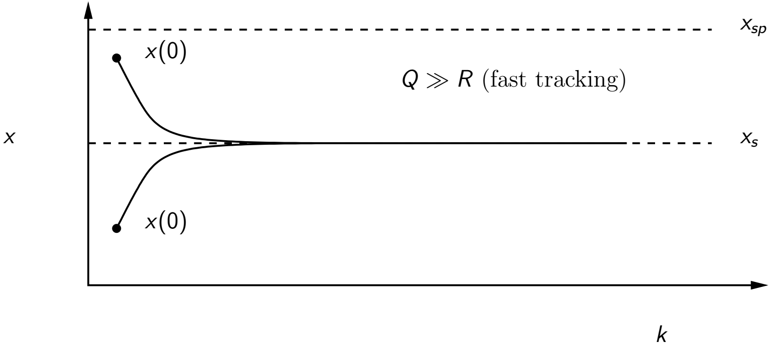
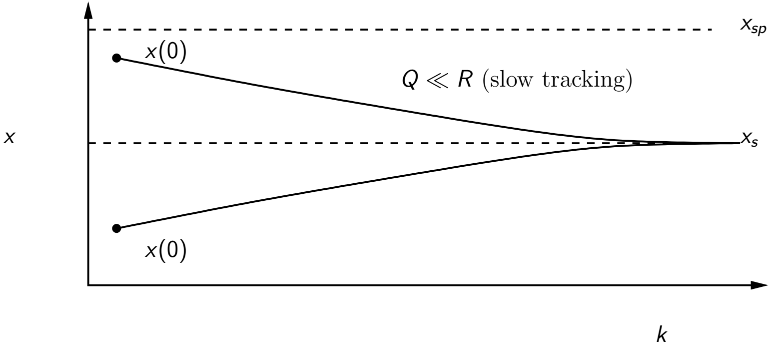
- Control law: $u^0(x) = \mathbf{u}^0(0; x)$

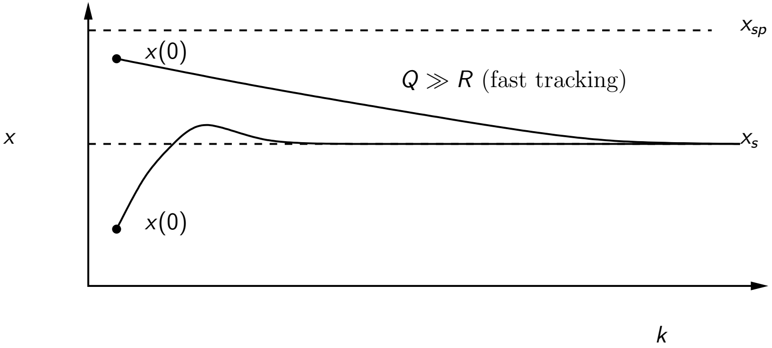
What closed-loop behavior is desirable? Fast tracking



What closed-loop behavior is desirable? Slow tracking



What closed-loop behavior is desirable? Asymmetric tracking



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- Simulations indicate the closed loop *is* stable
- How can we be sure?

Recent results for economic MPC

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- The economic MPC for linear dynamics, $f(x, u) = Ax + Bu$, and convex cost is asymptotically stabilizing (Rawlings, Bonné, Jørgensen, Venkat, and Jørgensen, 2008)
- Proof based on convexity of stage cost
- No Lyapunov function was found

Rotated cost and Lyapunov function

- Stability results extended to nonlinear systems satisfying strong duality. There exists constant $\lambda \in \mathbb{R}^n$ such that

$$\min_{(x,u)\in\mathbb{Z}} \ell(x,u) + \lambda'(x - f(x,u)) \geq \ell(x_s, u_s)$$

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- The Lyapunov function is based on rotated stage cost

$$L(x,u) = \ell(x,u) - \ell(x_s,u_s) + \lambda'(x - f(x,u))$$

- Diehl, Amrit, and Rawlings (2010)

Generalization to dissipative systems

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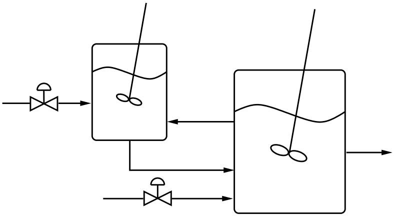
- Rotated stage cost

$$L(x,u) = \ell(x,u) - \ell(x_s,u_s) + \lambda(x) - \lambda(f(x,u))$$

- MPC also shown to outperform optimal periodic control
- Angeli, Amrit, and Rawlings (2010)

Nonlinear chemical reactor example

$$P_0 + B \longrightarrow P_1$$
$$P_1 + B \longrightarrow P_2$$



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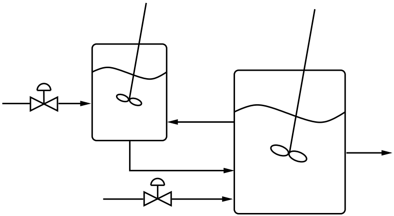

Species mass balances:

$$\dot{x}_1 = u_1 - x_1 - \sigma_1 x_1 x_2$$
$$\dot{x}_2 = u_2 - x_2 - \sigma_1 x_1 x_2 - \sigma_2 x_2 x_3$$
$$\dot{x}_3 = -x_3 + \sigma_1 x_1 x_2 - \sigma_2 x_2 x_3$$
$$\dot{x}_4 = -x_4 + \sigma_2 x_2 x_3$$

x_1, x_2, x_3, x_4: concentrations of P_0, B, P_1, P_2
u_1, u_2: inflow rates of P_0 and B

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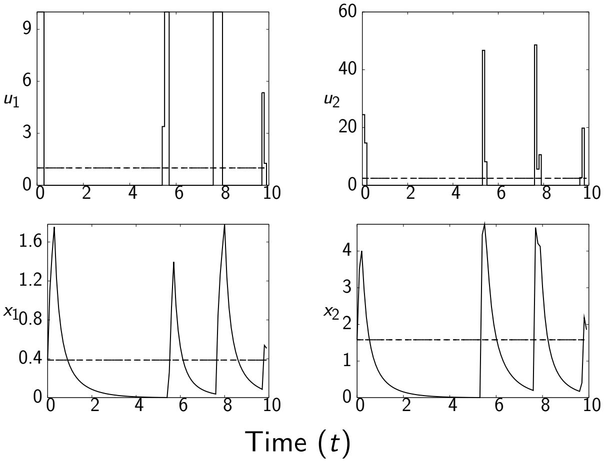

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  - Leverage from industrial implementation of linear MPC
  - This is a big opportunity!

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What do David Mayne and James Bond have in common?

Besides their dashing good looks . . . both turned 80 this year!

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> *SIR SEAN CONNERY told last night how he was "thrilled" that he had reached 80 - and promised that there was life in the old Bond yet.*

`www.dailyrecord.co.uk`
Aug 25 2010 Exclusive interview by Rick Fulton

Selected results from David Mayne's research

- Differential dynamic programming. Active participant in the control revolution of the 1960s. Jacobson and Mayne (1970)

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- Differential dynamic programming. Active participant in the control revolution of the 1960s. Jacobson and Mayne (1970)
- Particle filtering. First proponent of sampling as a way to address nonlinear estimation problems. Handschin and Mayne (1969)
- Stability of MPC. Helped unify the disparate stability results on receding horizon control and model predictive control. Papers and plenary lectures identifying the "ingredients" for closed-loop stability of MPC. Michalska and Mayne (1993); Mayne, Rawlings, Rao, and Scokaert (2000)

Selected results from David Mayne's research (cont.)

- Suboptimal MPC. How to ease the requirements for online solution of nonconvex optimizations so that the approach can be implemented in practice. Scokaert, Mayne, and Rawlings (1999)

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- Robust, tube-based MPC. When receding horizon is insufficient and why feedback is required. Mayne (1995); Langson, Chryssochoos, Raković, and Mayne (2004); Raković, Kerrigan, Kouramas, and Mayne (2005); Mayne, Seron, and Raković (2005)

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- Nonlinear moving horizon state estimation. Established the ingredients for stable nonlinear estimation. Rao, Rawlings, and Mayne (2003)

Book writing with David

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Writing a book is an adventure. To begin with it is a toy, then amusement, then it becomes a mistress, then it becomes a master, and then it becomes a tyrant,

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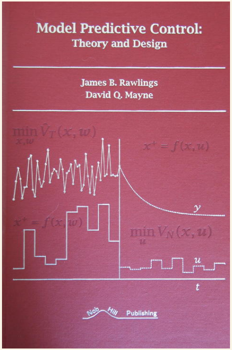
Writing a book is an adventure. To begin with it is a toy, then amusement, then it becomes a mistress, then it becomes a master, and then it becomes a tyrant, and the last phase is that just as you are about to become reconciled to your servitude, you kill the monster and strew him about to the public.

– Winston Churchill

Book writing with David



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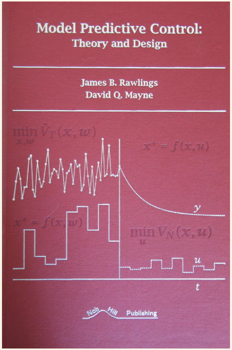

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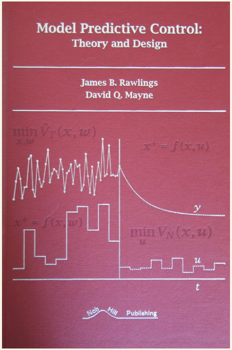

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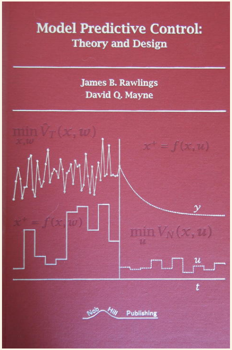

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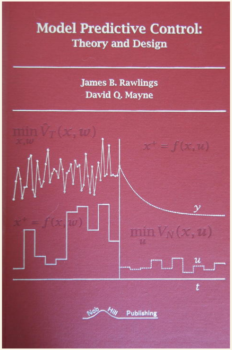

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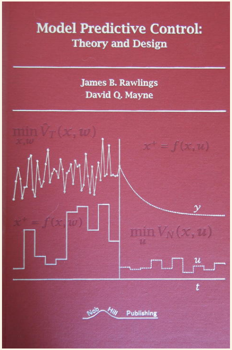
Book writing with David



- Published in 2009
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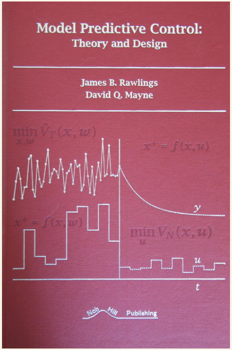

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Version	Date	Comment
1	2003 July 19	First version of book sources entered into version control

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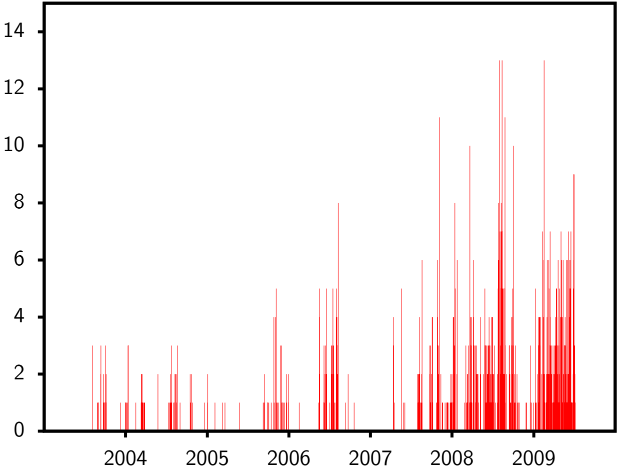
1182	2009 July 3	First printing completed

When are authors working on their books?



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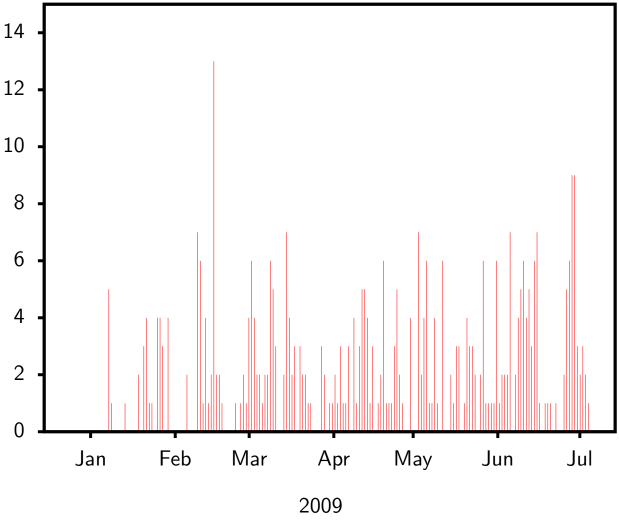

Personal observations about David

- High standards

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- High standards
- Dedication to the discipline

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- High standards
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- Love of the craft

Personal observations about David

- High standards
- Dedication to the discipline
- Love of the craft
- Generosity to colleagues

Further reading I

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